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## Smallest limited snakes

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### 1 Introduction

A (topological) disk is a subset of the euclidean plane homeomorphic to the unit ball. If two disks have a common interior point then we say that the disks overlap. A sequence  $\mathcal{C} = \langle C_1, \dots, C_n \rangle$  of mutually non overlapping congruent disks where  $C_i \cap C_j \neq \emptyset$  if and only if  $|i - j| \leq 1$  is called a snake. If the snake  $\mathcal{C}$  is not a proper subset of another snake of disks congruent to the members of  $\mathcal{C}$  then we say that the snake is limited.

We are concerned with the following question: What is the minimum number of mutually non overlapping congruent disks which can form a limited snake? Here we prove

**Theorem.** *The minimum number of mutually non overlapping congruent disks which can form a limited snake is four.*

Surprisingly, under the assumption of convexity the above problem seems to be much more complicated. Fig. 1 shows that six mutually non overlapping congruent copies of a

Auf einem Tisch legt man mit lauter gleichen Münzen eine „Münzschlange“: an eine erste Münze anstossend legt man eine zweite, daran anstossend eine dritte usw. Bei diesem Legespiel kann eine Konfiguration entstehen, bei der man weder am Kopf noch am Schwanz der Schlange eine weitere Münze anschliessen kann, weil der Platz durch andere Münzen des Schlangenkörpers versperrt wird. Welches ist die kleinste Anzahl Münzen, bei der dies vorkommen kann? Die Autoren untersuchen und beantworten die entsprechende Frage, wenn man die runden Münzen durch eine beliebige einfach zusammenhängende beschränkte Menge der Euklidischen Ebene ersetzt. Das entsprechende Problem für konvexe beschränkte Mengen ist hingegen noch offen.

certain rectangle can form a limited snake. Do there exist convex disks whose less than six mutually non overlapping congruent copies could form limited snakes? We conjecture that the answer to this question is in the negative.

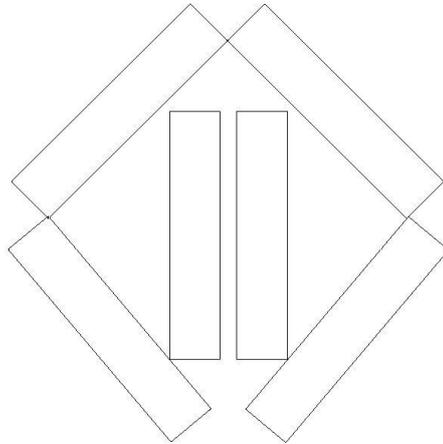


Fig. 1

Also, the problem of determining the minimum number of mutually non overlapping congruent copies of a given disk which can form a limited snake is very complicated. The only known result in this direction is that the minimum number of mutually non overlapping congruent balls which can form a limited snake is ten (see [1]).

For additional results on more restrictive variants of the snake problem, see [2, 3, 4, 5, 6, 7, 8].

## 2 Proof of the theorem

Fig. 2 shows that this minimum number is at most four.

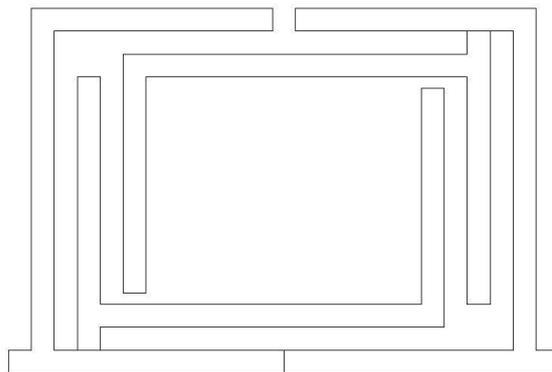


Fig. 2

To complete the proof we have to show that two or three mutually non overlapping congruent disks cannot form a limited snake. Let  $C$  be an arbitrary disk and let  $\mathcal{C} = \langle C_1, \dots, C_n \rangle$  be a limited snake consisting of disks congruent with  $C$ .

We start with the case  $n = 2$ . First assume that  $\text{conv } C_1 = \text{conv } C_2$ , i.e., the convex hulls of  $C_1$  and  $C_2$  coincide. If every boundary point of  $\text{conv } C_1$  belongs to  $C_1$ , i.e.,  $C_1$  is convex, then  $C_1$  and  $C_2$  coincide, which is impossible. Thus there exists a boundary point of  $\text{conv } C_1$  which does not belong to  $C_1$ . This point lies in the relative interior of a segment joining two extreme points, say  $A$  and  $B$ , of  $\text{conv } C_1$ . Recall that a point of a disk is an extreme point of the disk if there exists no segment in the disk that contains the point in its relative interior. The points  $A$  and  $B$  are extreme points of  $C_1$  and they can be joined with a path  $P_1$  whose points different from  $A$  and  $B$  lie in the interior of  $C_1$ . Also, the points  $A$  and  $B$  are extreme points of  $C_2$  and they can be joined with a path  $P_2$  whose points different from  $A$  and  $B$  lie in the interior of  $C_2$ . Then either the bounded region surrounded by  $P_1$  and the segment  $\overline{AB}$  contains  $P_2$  or the bounded region surrounded by  $P_2$  and the segment  $\overline{AB}$  contains  $P_1$ , which is impossible since  $\text{conv } C_1 = \text{conv } C_2$ .

Thus there exists a point  $P$  of  $C_1$  which does not belong to  $\text{conv } C_2$ . Then  $P$  can be strictly separated from  $C_2$  by a line  $l$ . Let  $l'$  be the support line of  $C_1$  which is parallel to  $l$  and does not separate  $C_1$  and  $C_2$ . Reflecting  $C_1$  with respect to  $l'$  we obtain a third copy of  $C$  which forms with  $C_1$  and  $C_2$  a three element snake, a contradiction.

Now we turn to the case  $n = 3$ . Let  $\overline{DE}$  be a diameter of  $C_1$  and consider the stripe  $S_1$  whose boundary lines, say  $l_1$  and  $l_2$ , go through  $D$  and  $E$ , respectively, and are perpendicular to  $\overline{DE}$ . If  $C_3$  is not contained in  $S_1$  then consider the support line  $l$  of  $C_3$  which is parallel to  $l_1$  and whose distance from  $S_1$  is maximal. Without loss of generality we may assume that  $l_2$  separates  $l$  and  $l_1$ . Let  $F$  be a common point of  $C_3$  and  $l$ . The disk  $C_2$  cannot intersect both  $l$  and  $l_1$  since the distance between the two lines is greater than the diameter of  $C_2$ . Thus either reflecting  $C_1$  with respect to  $l_1$  or reflecting  $C_3$  with respect to  $l$  we obtain a fourth copy of  $C$  which forms with  $C_1$ ,  $C_2$  and  $C_3$  a four element snake, a contradiction.

Thus  $C_3$  lies in  $S_1$ . Let  $\overline{GH}$  be a diameter of  $C_3$  and consider the stripe  $S_3$  whose boundary lines, say  $l_3$  and  $l_4$ , go through  $G$  and  $H$ , respectively, and are perpendicular to  $\overline{GH}$ . If  $C_1$  is not contained in  $S_3$  then repeating the previous argument we obtain a contradiction. Therefore  $C_1$  lies in  $S_3$ . If  $S_1 = S_3$ , i.e.,  $l_1 = l_3$  and  $l_2 = l_4$  without loss of generality, then  $D$  and  $G$  are different points since  $C_1 \cap C_3 = \emptyset$ . Now  $C_2$  does not contain both  $D$  and  $H$  since their distance is greater than the diameter of  $C_2$ . Therefore either reflecting  $C_1$  with respect to  $l_1$  or reflecting  $C_3$  with respect to  $l_2$  we obtain a fourth copy of  $C$  which forms with  $C_1$ ,  $C_2$  and  $C_3$  a four element snake, a contradiction. On the other hand, if  $S_1$  and  $S_3$  are different stripes then their intersection is a parallelogram which contains both  $C_1$  and  $C_3$ . The points  $D$  and  $E$  can be joined by a path  $P_3$  in  $C_1$  while  $G$  and  $H$  can be joined by a path  $P_4$  in  $C_3$ . Since the paths join opposite sides of the above parallelogram they necessarily intersect each other. But this is impossible since  $C_1$  and  $C_3$  are disjoint. This completes the proof of the theorem.

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