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Also set-valued functions do not like iterative roots

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1 Introduction

It seems that it is Ch. Babbage who first, yet at the beginning of the 19th century, wrote on iterative roots explicitly. Given a mapping $f : X \to X$ and a positive integer $n \geq 2$, the problem is to find a mapping $g : X \to X$ such that the n-th iterate of g, the composition g^n of n copies of g, is f, i.e., to solve the functional equation

$$g^n = f. \tag{1.1}$$

In [1] Babbage studied (1.1) for f being the identity mapping. After him a lot of results concerning the general case of (1.1) in various settings have been proved. Many of them can be found in the monographs [12] and [13] by M. Kuczma and M. Kuczma, B. Choczewski, R. Ger, respectively, as well as in the book [19] by Gy. Targonski. Some recent results have been presented in the survey papers [3] and [2].

Die Aufgabe, die n-te iterative Wurzel einer Abbildung $f : X \to X$ zu finden, besteht darin, eine Funktion $g : X \to X$ so zu bestimmen, dass $g^n = g \circ g \circ \ldots \circ g = f$ (n-fache Hintereinanderausführung) gilt. Für dieses Problem sind sowohl kombinatorische, als auch analytische Resultate bekannt. So besitzt beispielsweise $f : [0, 1] \to [0, 1]$, gegeben durch $f(x) = 4x(1 - x)$, keine iterative Wurzel. Die Autoren untersuchen in dieser Arbeit das analoge Problem für mengenwertige Abbildungen $f : X \to 2^X$. Es zeigt sich, dass selbst Monotonie- und Stetigkeitsannahmen, die bei gewöhnlichen Funktionen Existenz von Wurzeln sicher stellen, hierfür in diesem Fall im allgemeinen nicht ausreichen.

The purely combinatorial paper [10] by R. Isaacs gave a description of solutions to (1.1) for an arbitrary bijection f. The case of general f was completely solved by G. Zimmermann, Ph.D. student of Targonski, in her not well-known doctoral thesis [22] (see also [17] by G. Riggert noticing that *Zimmermann* is the maiden name of Riggert).

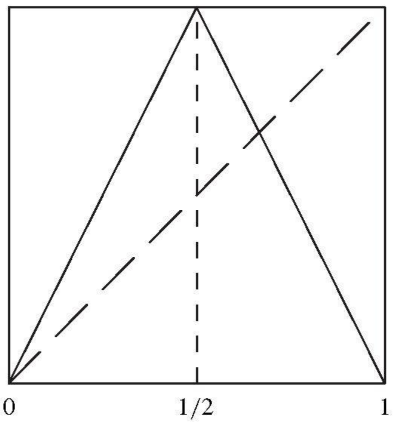

Fig. 1: Hat function

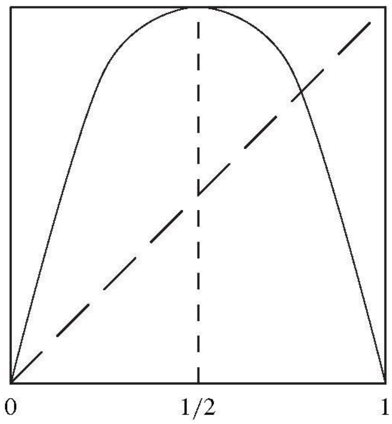

Fig. 2: Parabola $y = 4x(1 - x)$

It turns out that even very simple and nice functions can have no roots. For instance, this is the case if f is the so-called *hat function*, i.e. $f(x) = \min\{2x, 2 - 2x\}$ for $x \in [0, 1]$ (see Fig. 1) or f is the celebrated *parabola* $y = 4x(1-x)$ for $x \in [0, 1]$ (see Fig. 2). Of course, lack of roots for these functions can be deduced from Zimmermann's work. However, the reader surely can give a short and quite elementary argument in both cases. The above mentioned functions represent two important classes of mappings: piecewise monotone functions and polynomials. As follows, from [5], even in the class of piecewise monotone functions equation (1.1) leads to non-trivial questions. For f in this class sufficient conditions for nonexistence and existence of roots can be found in [21]. For polynomials the lack of roots is also a rather common phenomenon. A fundamental paper is [16] by R.E. Rice, B. Schweizer and A. Sklar, published in the *Monthly* almost 25 years ago. The answer to its title question "*When is* $f(f(z)) = az^2 + bz + c$?" is *never*. Similar results concerning some other polynomials can be found in [7] and [8]. Nonexistence of roots, both formal and holomorphic, was indicated by S. Bogatyi in his important article [6].

Difficulties appearing when solving equation (1.1), even in the class of continuous monotone self-mappings of an interval, have been enlightened in the crucial paper [9] by P.D. Humke and M. Laczkovich. Roughly speaking, they proved that the set of functions having a root is an analytic but non-Borel subset of the space $C([0, 1], \mathbb{R})$ endowed with the sup-norm. The papers [18] and [4] by K. Simon and A. Blokh, respectively, show that this set is small in $C([0, 1], [0, 1])$ both from the category (see [18, 4]) and measure-theoretical (cf. [18]) points of view. Nonexistence of roots is typical also for some regular functions (see [20]).

Recently some natural ideas of using set-valued functions have been examined (cf., for instance, [14, 15, 11]). One can consider replacing single-valued functions by set-valued functions in (1.1) both for f and g. It seems that up to now there are no notions leading to a satisfactory result in such a case.

In this paper we show that the phenomenon of lack of iterative roots appears also when studying some set-valued functions with exactly one value not being a singleton. Even imposing assumptions like continuity or strict monotonicity on the "single-valued parts" of such a set-valued function does not guarantee the existence of its square roots (see Example 2). This shows that maybe the situation for set-valued functions is even more sophisticated since those assumptions usually allow us to find roots in the single-valued case.

2 Main results

Given a set-valued function $f : X \to 2^Y$, the image $f(A)$ of a set $A \subset X$ is defined by

$$f(A) = \bigcup_{x \in A} f(x).$$

Then we can introduce the composition $g \circ f$ of set-valued functions $f : X \to 2^Y$ and $g : Y \to 2^Z$ by the familiar formula

$$(g \circ f)(x) = g(f(x)).$$

Clearly this operation is associative. So, for every positive integer n, we can define the n-th iterate of $g : X \to 2^X$ as the composition of n copies of g:

$$g^n = \underbrace{g \circ \ldots \circ g}_{n-\text{times}}.$$

Consequently, the problem of looking for solutions $g : X \to 2^X$ to (1.1) for set-valued functions f is posed in a proper way.

Remark that *if $g : X \to 2^X$ is an iterative root of $f : X \to 2^X$ then f and g commute, i.e. $f \circ g = g \circ f$.* In fact, assume that $g^k = f$ for a positive integer k and fix an $x \in X$. If $z \in f(g(x))$ then $z \in f(y)$ for a $y \in g(x)$, that is, $z \in g^k(g(x)) = g(g^k(x)) = g(f(x))$. Conversely, if $z \in g(f(x))$ then $z \in g(y)$ for a $y \in f(x)$, so $z \in g(f(x)) = g(g^k(x)) = g^k(g(x)) = f(g(x))$.

In what follows, we consider X as an arbitrary set and let $\#A$ denote the cardinality of a subset $A \subset X$.

Proposition. *Consider a set-valued function $f : X \to 2^X$ and let $g : X \to 2^X$ be its iterative square root. If there is a point $c \in X$ such that*

(i) *$\#f(x) = 1$ for every $x \in X \setminus \{c\}$ and*

(ii) *$f(x_0) = \{c\}$ for an $x_0 \in X$,*

then $\#g(c) \leq 1$.

Proof. Suppose that

$$\#g(c) \geq 2. \tag{2.2}$$

It follows from (i) that g has non-void values only. Fix a $p \in g(x_0)$. Then

$$g(p) \subset g(g(x_0)) = f(x_0) = \{c\},$$

that is in fact $g(p) = \{c\}$, whence

$$f(p) = g(g(p)) = g(c).$$

Therefore, by (2.2) and (i), we get $p = c$. Thus, we have proved that $g(x_0) = \{c\}$, which gives

$$g(c) = g^2(x_0) = f(x_0) = \{c\},$$

a contradiction to (2.2). $\square$

Our main results are simple consequences of the proposition.

Theorem 1. *Let $f : X \to 2^X$. If there are a point $c \in X$ and a positive integer n such that* (i) *and* (ii) *hold,*

(iii) $\# f(c) > n$, *and*

(iv) $\#\{x \in X : f(x) = \{y\}\} \leq n$ *for every* $y \in X$,

then f has no iterative square roots.

Proof. Suppose that f has a square root $g : X \to 2^X$. By (i) and (iii) all the values of f and consequently of g are non-void.

Firstly, we claim that

$$\# g(x) \leq n \qquad \text{for} \quad x \in X \backslash \{c\}. \tag{2.3}$$

In order to see this, fix an $x \in X \backslash \{c\}$. Take any $v \in g(x)$. Since $f(x)$ is a singleton and

$$f(x) = g(g(x)) = \bigcup_{u \in g(x)} g(u),$$

for every $u \in g(x)$ we have $g(u) = g(v)$, whence $f(u) = f(v)$. This gives the inclusion

$$g(x) \subset \{u \in X : f(u) = f(v)\}. \tag{2.4}$$

If $g(x)$ is not a singleton then, according to (2.4) and (i), $f(v)$ is a singleton, whence using (2.4) again and (iv) we complete the proof of (2.3).

Since all the values of g are non-void, it follows from the proposition that $g(c) = \{u\}$ with a $u \in X$. Then $g(u) = g^2(c) = f(c)$, whence, by (iii), we have $\# g(u) > n$. So, $u \neq c$, which contradicts (2.3). $\square$

Theorem 2. *Let $f : X \to 2^X$. If there is a point $c \in X$ such that* (i) *and* (ii) *hold,*

(v) $\# f(c) > 1$, *and*

(vi) $c \in f(c)$,

then f has no iterative square roots.

Proof. Suppose that f has a square root $g : X \to 2^X$. By (i) and (v) all the values of g are non-void. Therefore, it follows from the proposition that $g(c) = \{u\}$ for a $u \in X$.

Then $g(u) = g^2(c) = f(c)$, whence, by (v), we have $\#g(u) > 1$, implying that $u \neq c$. On account of (v) the set $g(u)$ contains a point $v \in X\backslash\{c\}$. Moreover, according to (vi), $c \in f(c) = g(u)$. Therefore, since

$$g(v) \cup g(c) \subset g(g(u)) = f(u),$$

(i) gives $g(v) = g(c)$. Consequently, $f(v) = f(c)$, which contradicts (i) and (v). □

3 Examples

1. Consider $f : [0, 1] \to 2^{[0,1]}$ given by

$$f(x) = \begin{cases} \frac{3}{2}x, & \text{if } x \in [0, \frac{1}{2}), \\ [\frac{1}{2}, \frac{3}{4}], & \text{if } x = \frac{1}{2}, \\ x, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Then assumptions (i)–(iv) in Theorem 1 are satisfied with $c = 1/2$ and $n = 2$. Consequently, f has no square root.

2. There are some properties, like e.g. strict monotonicity, continuity, lack of fixed points of f in the interior of its interval domain, which guarantee the existence of iterative roots of single-valued functions (cf., e.g., [12, Chap. XV] and [13, Chap. 11]). For set-valued functions the situation is more complicated, which can be seen by considering $f_1 : [0, 1] \to 2^{[0,1]}$ and $f_2 : [0, 1] \to 2^{[0,1]}$ defined by

$$f_1(x) = \begin{cases} \frac{5}{3}x, & \text{if } x \in [0, \frac{1}{2}), \\ [\frac{2}{3}, \frac{5}{6}], & \text{if } x = \frac{1}{2}, \\ \frac{2}{3}(x-1)+1, & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

$$f_2(x) = \begin{cases} \frac{4}{3}x, & \text{if } x \in [0, \frac{1}{2}), \\ [\frac{2}{3}, \frac{5}{6}], & \text{if } x = \frac{1}{2}, \\ \frac{1}{3}(x-1)+1, & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

respectively. Both of them are upper semicontinuous and have no fixed points in $(0, 1)$. Moreover, $f_1|_{[0,1/2)}$ and $f_1|_{(1/2,1]}$ are both strictly increasing and the (single-valued) function $f_1|_{[0,1]\backslash\{1/2\}}$ is continuous. For f_2 we have even more: $f_2|_{[0,1]\backslash\{1/2\}}$ is strictly increasing and continuous. Nevertheless, by Theorem 1, where we take $c = 1/2$ and $n = 3 - j$ for f_j $(j = 1, 2)$, both f_1 and f_2 have no square roots. Observe also that $1/2 \notin f_1(1/2)$ and $1/2 \notin f_2(1/2)$, that is, condition (vi) is not satisfied. Consequently, Theorem 1 does not follow from Theorem 2.

3. In the case of f_3 as shown in Fig. 5 we have no roots again, as observed for $n = 4$.

4. The shape of the graph of f_4 (see Fig. 6), given by

$$f_4(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \in [0, \frac{1}{2}), \\ [\frac{1}{4}, \frac{3}{4}], & \text{if } x = \frac{1}{2}, \\ \frac{1}{2}(x-1)+1, & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

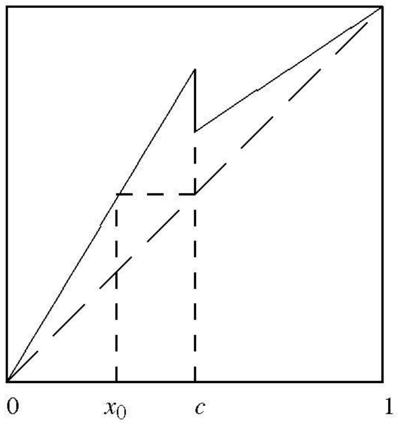

Fig. 3: f_1 with $n = 2$

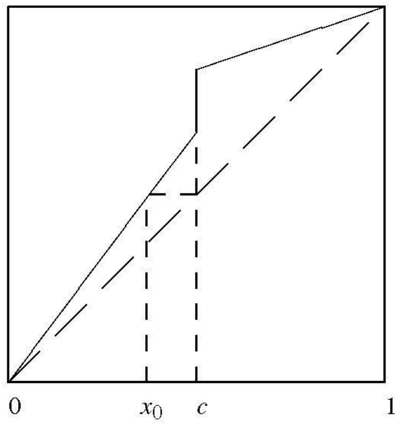

Fig. 4: f_2 with $n = 1$

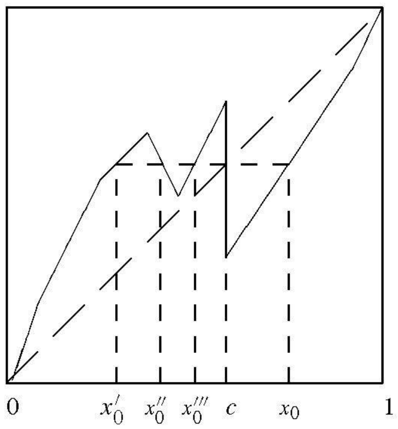

Fig. 5: f_3 with $n = 4$

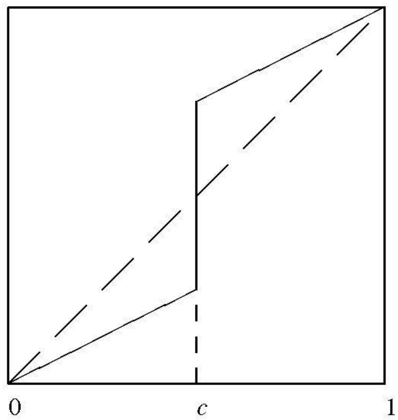

Fig. 6: f_4 has roots

is similar to the graph of f_2, but f_4 has a square root. One can easily verify that $g : [0, 1] \to 2^{[0,1]}$, defined by

$$g(x) = \begin{cases} \frac{1}{\sqrt{2}}x, & \text{if } x \in [0, \frac{1}{2}), \\ [\frac{1}{2\sqrt{2}}, 1 - \frac{1}{2\sqrt{2}}], & \text{if } x = \frac{1}{2}, \\ \frac{1}{\sqrt{2}}(x-1) + 1, & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

satisfies $g^2 = f$. Observe, however, that Condition (ii) fails, where c has to be $1/2$.

5. Consider the set-valued functions $f_5 : [0, 1] \to 2^{[0,1]}$ and $f_6 : [0, 1] \to 2^{[0,1]}$ defined by

$$f_5(x) = \begin{cases} \frac{3}{2}x, & \text{if } x \in [0, \frac{1}{2}), \\ \{\frac{1}{2}, \frac{3}{4}\}, & \text{if } x = \frac{1}{2}, \\ x, & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

$$f_6(x) = \begin{cases} \frac{3}{2}x, & \text{if } x \in [0, \frac{1}{2}), \\ [\frac{1}{2}, \frac{3}{4}], & \text{if } x = \frac{1}{2}, \\ \frac{1}{2}, & \text{if } x \in (\frac{1}{2}, \frac{3}{4}], \\ 2(x-1)+1, & \text{if } x \in (\frac{3}{4}, 1], \end{cases}$$

respectively. Condition (iii) is not satisfied by f_5 since $c = 1/2, n = 2$ and $\#f_5(c) = 2$. For f_6 condition (iii) is not satisfied because $c = 1/2, n = \aleph_0$ and $\#f_6(c) = \aleph_0$. However, they both satisfy (v) and (vi). Theorem 2 shows that none of them has a square root. Consequently, this also implies that Theorem 2 does not follow from Theorem 1.

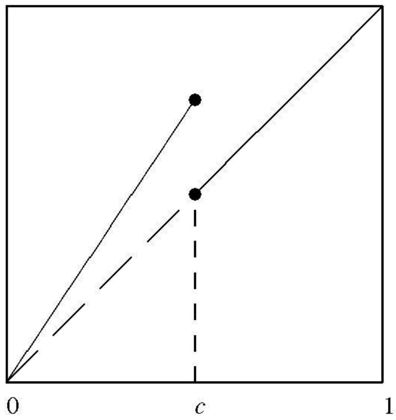

Fig. 7: f_5 with $n = 2$

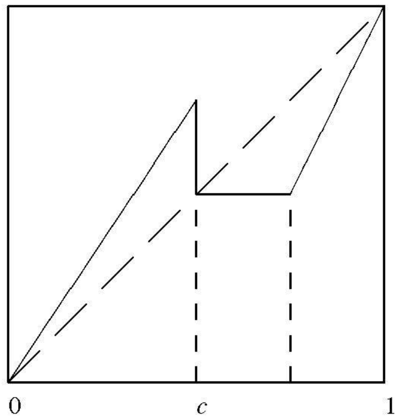

Fig. 8: f_6 with $n = \infty$

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