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## Elemente der Mathematik

# On a result of James and Niven concerning unique factorization in congruence semigroups

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The theory of non-unique factorizations in integral domains and monoids is a very active area of current research (see both [1] and [4] to view recent trends in this work). To demonstrate the phenomena of non-unique factorizations, we consider a result from the classical setting on uniqueness of factorizations by James and Niven [11]. We proceed as follows: Let  $\mathbb{N}$  represent the natural numbers and suppose that  $M \subseteq \mathbb{N}$  is a multiplicative semigroup. M is called a *congruence semigroup* if there exists a natural number n such

Im Hilbertschen Monoid  $1+4\mathbb{N}_0=\{1,5,9,13,\ldots\}$  ( $\mathbb{N}_0=$  natürliche Zahlen inklusive Null) ist die Zerlegung in irreduzible Faktoren nicht eindeutig: Es gilt zum Beispiel  $441=9\cdot 49=21\cdot 21$ . Hilberts Monoid ist ein Beispiel einer Kongruenz-Halbgruppe. Ein klassisches Resultat von James und Niven besagt, dass in einer Kongruenz-Halbgruppe M genau dann der Fundamentalsatz der Arithmetik gilt, wenn M aus allen Zahlen besteht, die relativ prim zu einer festen Zahl  $n\in\mathbb{N}$  sind. Die Autoren der vorliegenden Arbeit untersuchen das andere Extrem, nämlich den Fall, wo M aus allen Zahlen besteht, die nicht relativ prim zu einer festen Zahl  $n\in\mathbb{N}$  sind. Sie zeigen, dass in diesem Fall wenigstens die Anzahl der Primfaktoren bei der Zerlegung einer Zahl eindeutig ist.

that

$$x \in M$$
 and  $x \equiv y \pmod{n}$  for  $y \in \mathbb{N}$  implies  $y \in M$ .

If M is as above, then we call n a modulus of definition of M. It follows directly from the definition that a congruence semigroup M of modulus n is completely determined by n and  $M \cap \{1, 2, ..., n\}$ . In a congruence semigroup M, we call an element x irreducible if x cannot be written in the form yz where y and z are nonunits of M (note that M possesses at most one unit, that being 1). The classic proof that all natural numbers can be factored as a product of primes can be easily modified to show that each nonunit of a congruence semigroup can be factored as a product of irreducible elements. In general, such a semigroup is called atomic. The interested reader can find more information on congruence semigroups in [8] and a review of basic algebra terminology in [10].

Examples of congruence semigroups can be found throughout the mathematical literature. In particular, Davenport [7, p. 21] uses the "Hilbert monoid"

$$1 + 4\mathbb{N}_0 = \{1, 5, 9, 13, 17, 21, \ldots\}$$

as an example of a multiplicative system where the Fundamental Theorem of Arithmetic fails. To be precise, in this system,

$$441 = 21 \cdot 21 = 9 \cdot 49$$

and 9, 21 and 49 are all nonassociated irreducibles in  $1 + 4N_0$ .

Hence, it is reasonable to ask which congruence semigroups do satisfy the Fundamental Theorem of Arithmetic. This question was answered by James and Niven in [11], where they prove the following interesting result. We will require the following notation: if  $n \in \mathbb{N}$ , then set

$$A(n) = \{ m \mid m \in \mathbb{N} \text{ and } \gcd(m, n) = 1 \}$$

and 
$$B(n) = \mathbb{N} - A(n)$$
.

**Theorem (James and Niven [11]).** Let M be a congruence semigroup. M has unique factorization of elements into products of irreducible elements if and only if there exists a positive integer n with  $M \cap A(n) = A(n)$  and  $M \cap B(n) = \emptyset$ . In other words, M has unique factorization if and only if M consists of all elements relatively prime to a fixed positive integer n.

An alternate proof of this theorem due to Halter-Koch (which uses the *divisor theory* of a commutative cancellative monoid) can be found in [9]. As a byproduct of the theorem, we point out that the modulus for a congruence semigroup is not unique. Notice that letting n=2 or 4 in the theorem produces the same semigroup. Hence, this M can be viewed with modulus of definition 2 or 4. While the modulus is not unique, it is obvious that each congruence semigroup has a unique *minimal modulus*.

We are struck by what happens in the other extreme suggested by the theorem (i.e., when M consists of all elements not relatively prime to a fixed positive integer n). It turns out that such an M also exhibits an interesting factorization property.

**Proposition.** Let  $n = p_1^{n_1} \cdots p_k^{n_k}$  be a positive integer where the  $p_i$ 's are distinct primes and the  $n_i$ 's positive integers. Set

$$M = \{m \in \mathbb{N} \mid \gcd(m, n) \neq 1\}.$$

M is a congruence semigroup with minimal modulus  $n' = p_1 \cdots p_k$  which satisfies the following factorization property: If  $x \in M$  and

$$x = y_1 \cdots y_s = z_1 \cdots z_t \tag{*}$$

where each  $y_i$  and  $z_j$  is irreducible in M, then s = t.

*Proof.* Since the product of two numbers not relatively prime to n is again not relatively prime to n, M is closed under multiplication and is a multiplicative semigroup. It follows directly from the hypothesis of the proposition and elementary number theory that M is a congruence semigroup of modulus n. We show that M also has modulus  $n' = p_1 \cdots p_k$ . Setting

$$M' = \{ m \in \mathbb{N} \mid \gcd(m, n') \neq 1 \}$$

we obtain, as above, that M' is a congruence monoid of modulus n'. For  $m \in \mathbb{N}$  it follows that  $\gcd(m,n) \neq 1$  if and only if  $\gcd(m,n') \neq 1$ . Hence M=M' and n' is a modulus of definition for M. We argue that this is the minimal modulus. Suppose M is defined by some modulus d < n'. Then there exists an i such that  $p_i \nmid d$ . Now, by definition  $p_i \in M$ , but note that  $p_i^{\varphi(d)} \equiv 1 \pmod{d}$  (where  $\varphi$  represents the Euler  $\varphi$ -function), and hence  $1 \in M$ , a contradiction.

We now show that M satisfies (\*). By the definition of M, if  $x \in M$ , then  $x = p_1^{\alpha_1} \cdots p_k^{\alpha_k} w$  where the  $\alpha_i$ 's are nonnegative integers (with at least one nonzero) and  $w \in \mathbb{N}$  with  $\gcd(w, n') = 1$ . Define a function  $f: M \to \mathbb{N}$  by

$$f(x) = \sum_{i=1}^{k} \alpha_i.$$

It is easy to verify that for x and  $y \in M$  we have f(xy) = f(x) + f(y).

Claim:  $x \in M$  is irreducible in M if and only if f(x) = 1.

*Proof of Claim:* ( $\Rightarrow$ ) Suppose that  $x \in M$  and f(x) > 1. Write x = pqk where p and q are not necessarily distinct primes which divide n' and  $k \in \mathbb{N}$ . By definition, p, q and qk are in M. Hence pqk = (p)(qk) and thus is not irreducible in M.

( $\Leftarrow$ ) Suppose x = pk where p is a prime divisor of n' and gcd(k, n') = 1. If x = yz where y and  $z \in M$ , then 1 = f(x) = f(yz) = f(y) + f(z), which implies that either f(y) or f(z) = 0, a contradiction.

Now, suppose that  $x \in M$  and

$$x = y_1 \cdots y_s = z_1 \cdots z_t$$

where each  $y_i$  and  $z_j$  is irreducible in M. Then  $f(x) = \sum_{i=1}^{s} f(y_i) = \sum_{i=1}^{t} f(z_i)$ . Since each  $f(y_i) = 1 = f(z_i)$ , we have that f(x) = s = t and the result follows.

We close with some comments concerning the proposition.

- (1) An atomic semigroup (or monoid) which satisfies (\*) is called *half-factorial*. More information on the half-factorial property can be found in [5].
- (2) The James and Niven result indicates that the set of odd integers, when viewed as a semigroup, has unique factorization. On the other hand, the proposition indicates that the set of even integers is half-factorial. As the proof indicates, a non-unique factorization in the set of even integers is given by

$$6 \cdot 10 = 2 \cdot 30,$$

where 2, 6, 10 and 30 are all irreducible as even integers.

- (3) Not all half-factorial congruence semigroups are of the form M in the proposition. The Hilbert Monoid,  $H = 1 + 4\mathbb{N}_0$  is also half-factorial. To see this, notice that  $x \in H$  is irreducible if and only if
  - (i) x is prime in  $\mathbb{N}$ , or
  - (ii)  $x = q_1q_2$  where  $q_1$  and  $q_2$  are not necessarily distinct primes in  $\mathbb{N}$  which are congruent to 3 (mod 4).

Hence, if  $x \in H$  is of the form

$$x = p_1 \cdots p_s q_1 \cdots q_t$$

where each  $p_i$  is a prime congruent to 1 (mod 4) and each  $q_j$  is a prime congruent to 3 (mod 4), then any irreducible factorization of x in H has length  $s + \frac{t}{2}$  (note that t will necessarily be even). Half-factorial congruence semigroups which are also arithmetic sequences have been characterized in [3, Theorem 2.6].

(4) To exhibit a congruence semigroup which is neither factorial nor half-factorial, let  $M = 1 + 5\mathbb{N}_0$ . In M we have

$$81 \cdot 2401 = 21 \cdot 21 \cdot 21 \cdot 21$$

and each of 81, 2401 and 21 are irreducible in M. A good general reference on monoids which do not satisfy the unique factorization property is [6].

(5) The function f in the proof of the proposition is known as a *semi-length function* on M. The reader can find more information on semi-length functions in [2].

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