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Extreme values for the area of rectangles with vertices on concentric circles

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1 Introduction

In [3], the following unusual geometrical problem was proposed:

Problem 1. *Consider three positive numbers x , y and z such that there exists a rectangle $ABCD$ with the property that for some point P in the interior of the rectangle, $PA = x$, $PB = y$ and $PC = z$. Find the maximum possible area of a rectangle with this property in terms of x , y , z .*

We are going to consider also the problem of finding the minimum area of such a rectangle, where P is not necessarily in the interior of the rectangle. For example, if $x = 7$, $y = 5$

Im nachfolgenden Beitrag bestätigen die Autoren zunächst die Behauptung, dass die Bedingung $y^2 + t^2 = x^2 + z^2$ an die Radien x , y , z , t von vier konzentrischen Kreisen notwendig und hinreichend für die Existenz eines Rechtecks mit je einer Ecke auf je einem der vier Kreise ist. Abgesehen von entarteten Fällen stellen die Autoren überdies fest, dass es in der Regel jeweils unendlich viele Lösungsmöglichkeiten gibt, und sie stellen sich die Aufgabe, das Rechteck mit dem maximalen/minimalen Flächeninhalt zu finden. Es gelingt ihnen, dieses Optimierungsproblem mit Hilfe von Lagrange-Multiplikatoren explizit zu lösen. Es sei darauf hingewiesen, dass eine verwandte Aufgabenstellung kürzlich im American Mathematical Monthly Erwähnung fand (siehe [3]).

and $z = 1$, the maximum area is 32 and the minimum area is 18 (P not necessarily in the interior). We find a surprisingly simple formula for the maximum and minimum area using various tools that range from geometry and calculus to elementary number theory. Further, we show that the example given is one of the infinitely many instances when the extreme values for the area and the data are integers.

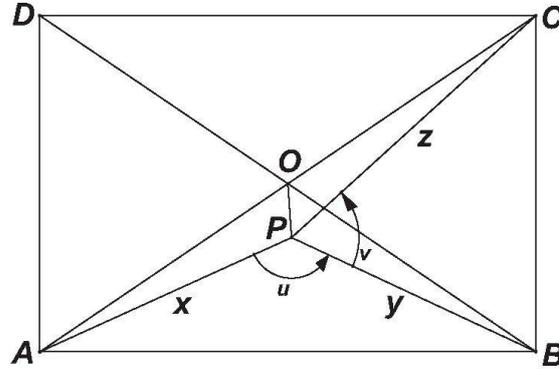


Fig. 1

Let us observe that there are some restrictions that one would have to impose on x , y and z in order for a rectangle with the required properties to exist. Let O be the intersection of the diagonals \overline{BD} and \overline{AC} . Using the formula relating the median to the side lengths of the triangle we get $4PO^2 = 2(x^2 + z^2) - AC^2 = 2(t^2 + y^2) - BD^2$, where $t = PD$. Thus, $t = \sqrt{x^2 + z^2 - y^2}$, so we need to have

$$y < \sqrt{x^2 + z^2}. \tag{1}$$

This condition is also sufficient for the existence of a rectangle $ABCD$ if the point P is not required to be necessarily in the interior of the rectangle with $PA = x$, $PB = y$ and $PC = z$, as we shall show in the next section. Hence, we may consider the slightly modified and in some respect more general problem:

Problem 2. *Given four concentric circles $C(P, x)$, $C(P, y)$, $C(P, z)$ and $C(P, t)$ of center P and radii x, y, z, t , resp., where $x^2 + z^2 = y^2 + t^2$, consider four points, $A \in C(P, x)$, $B \in C(P, y)$, $C \in C(P, z)$ and $D \in C(P, t)$, such that the quadrilateral $ABCD$ is a rectangle (in general, there may be two such rectangles for each particular position of A and B – see Fig. 2). Find the extreme values for the area of such a rectangle.*

Problem 1 follows from Problem 2 since the maximum is attained in a position where P is in the interior of the rectangle as shown in Section 2. Let us observe that Problem 2 is equivalent to the following:

Problem 3. *Given three positive numbers x, y, z such that $y < \sqrt{x^2 + z^2}$, determine the extreme values of the area of a right triangle $\triangle ABC$ ($\angle ABC = \pi/2$) with $A \in C(P, x)$, $B \in C(P, y)$, $C \in C(P, z)$.*

An advantage of formulating the problem this way is that one can easily see that the lengths of the rectangle’s sides are bounded quantities. Therefore, the maximum and minimum

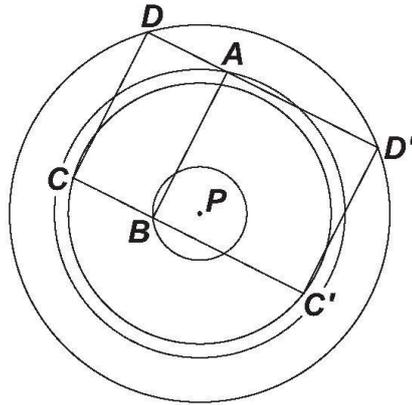


Fig. 2

area should be positive numbers (it is easy to see that the minimum is zero if and only if $y = x$ or $y = z$).

Problem 1 is unusual because of its surprisingly simple answer in spite of our rather laborious solution.

Theorem 1.1. *The maximum area of the rectangle in Problem 1 is equal to $xz + yt$ and the minimum area is $|xz - yt|$.*

2 Solution and commentaries

Our strategy to solve Problem 1 is a classical one. First of all we are mainly concerned with Problem 3. Referring to Fig. 1, we introduce parameters u and v which are oriented angles $u = m(\angle APB)$ and $v = m(\angle BPC)$ (angles rotated clockwise are considered negative and angles rotated counterclockwise are considered positive) and so $u, v \in (-\pi, \pi]$. All other angles are considered as in elementary Euclidean geometry: angles between 0 and π . We write the problem as an optimization of a function of two variables with a constraint (since our problem has essentially one degree of freedom). We apply the Lagrange multipliers method to find the associated critical points, and finally, we identify the points that indeed give the maximum or the minimum for our objective function.

Proposition 2.1 (Objective and Constraint Function). *The area of the rectangle $ABCD$ is twice the area of the triangle $\triangle ABC$ and as a function of u and v it is given by the formula*

$$\text{Area}(ABCD) = |xy \sin u + yz \sin v - xz \sin(u + v)|, \quad (2)$$

where P is not necessarily in the interior of the rectangle. The constraint equation is

$$xy \cos u + yz \cos v - xz \cos(u + v) - y^2 = 0. \quad (3)$$

Proof. We use vector calculus to derive $\text{Area}(ABCD) = 2 \text{Area}(\triangle ABC) = 2|\vec{AB} \times \vec{BC}| = |(\vec{PB} - \vec{PA}) \times (\vec{PC} - \vec{PB})|$. Further, $\text{Area}(ABCD) = 2|\vec{PA} \times \vec{PB} + \vec{PB} \times$

$|\vec{PC} - \vec{PA} \times \vec{PC}| = |xy \sin u + yz \sin v - xz \sin(u + v)|$. For the constraint equation we employ the notion of inner product of two vectors as opposed to the cross product that appeared in the derivation of (2). The constraint is given by the fact that $\triangle ABC$ is a right triangle at B . This can be equivalently written as $AB^2 + BC^2 = AC^2$. This can be written as $(\vec{PB} - \vec{PA})^2 + (\vec{PC} - \vec{PB})^2 = (\vec{PC} - \vec{PA})^2$. After simplifications this can be reduced to (3). \square

Let us reduce the number of given data by introducing $a = \frac{y}{z}$ and $b = \frac{y}{x}$. The inequality (1) is equivalent to $a^2 b^2 < a^2 + b^2$. So, Problem 3 is rephrased as

$$\left. \begin{array}{l} \text{Maximize/Minimize } f(u, v) = a \sin u + b \sin v - ab \sin(u + v) \\ \text{under the constraint} \\ g(u, v) := a \cos u + b \cos v - \cos(u + v) - ab, \\ (u, v) \in (-\pi, \pi] \times (-\pi, \pi]. \end{array} \right\} \quad (4)$$

This analysis problem is interesting in itself because of the symmetry of the two functions involved and of its relatively simple answer. We would like to make more precise the domain \mathcal{D} of u which is determined by the existence of a right triangle $\triangle ABC$, with $A \in C(P, x)$, $B \in C(P, y)$, $C \in C(P, z)$ having $\angle APB = u$ and a negative oriented angle $\angle ABC = -\pi/2$. We call this the positive orientation of $\triangle ABC$. In the parts (b) and (c) of the next proposition, y is assumed to satisfy (1).

Proposition 2.2.

- (a) Condition (1) is necessary and sufficient for the existence of $\triangle ABC$.
- (b) If $y \leq z$, then \mathcal{D} is $(-\pi, \pi]$.
- (c) If $y > z$, then $\mathcal{D} = [\arccos(\frac{y^2 - z^2 + tz}{xy}), \arccos(\frac{y^2 - z^2 - tz}{xy})]$.

Proof. The necessity of (1) was already shown, and the sufficiency is a consequence of (b) and (c). First, if $y \leq z$, the circle $C(P, y)$ is inside of or equal to $C(P, z)$ and so, the perpendicular on \overline{AB} at B is going to intersect the circle $C(P, z)$ at one point (if $y = z$), or at two points. We choose C to be one of these points in such a way that the orientation of the triangle is positive for every $u \in \mathcal{D}$. Hence, since this is possible for every particular position of B (with A fixed), (b) follows.

We assume now that $y > z$. Thus, $C(P, z)$ is in the interior of $C(P, y)$, and so the perpendicular on \overline{AB} at B is not going to intersect the circle $C(P, z)$ unless the angle u is in the right range. This range is given by the fact that AB must be limited in length in the following way. Applying the triangle inequality in $\triangle PCD$ (see Fig. 3), we see that $|z - t| \leq AB \leq z + t$. Equivalently, using the law of cosines in $\triangle APB$, we obtain $(t - z)^2 \leq x^2 + y^2 - 2xy \cos(u) \leq (z + t)^2$. Therefore, $\frac{y^2 - z^2 - tz}{xy} \leq \cos(u) \leq \frac{y^2 - z^2 + tz}{xy}$. Thus, $u \in \mathcal{D}$ or $u \in -\mathcal{D}$. Every angle $u \in -\mathcal{D}$ has to be excluded because it makes $\angle ABC$ positive.

Now, if $u \in \mathcal{D}$, then we construct $\triangle PCD$ with $C \in C(P, z)$, $D \in C(P, t)$, and \overline{CD} parallel and of the same length as \overline{AB} . This is possible because $u \in \mathcal{D}$ implies that a triangle

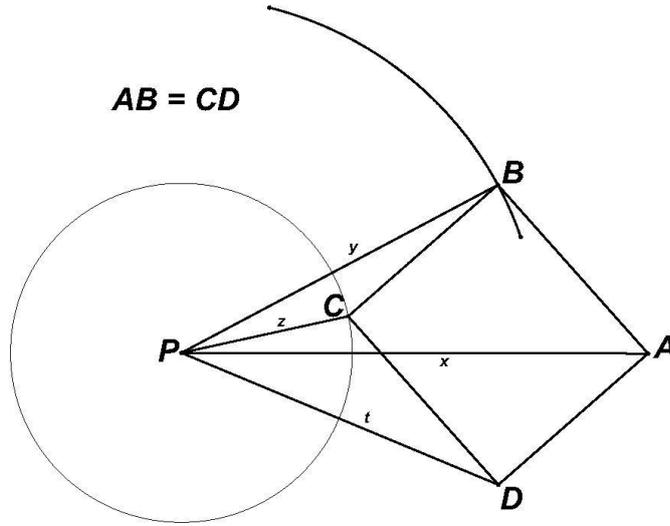


Fig. 3

congruent to $\triangle PCD$ exists and then one rotates C until the side \overline{CD} becomes parallel to \overline{AB} . Hence, the quadrilateral $ABCD$ is a parallelogram. The fact that $ABCD$ is a rectangle follows from the equality $x^2 + z^2 = y^2 + t^2$ since this implies that the diagonals of $ABCD$ are equal using an argument as the one used to derive condition (1). The orientation of $\triangle ABC$ is positive because $u \in \mathcal{D}$ and $C(P, z)$ is in the interior of $C(P, y)$. \square

Remark. In case (c) we consider both solutions for C . This means that v is going to have two solutions in general (except for the endpoints of the interval) for each value of $u \in \mathcal{D}$. We do not lose any generality when looking for the extreme values because for every value of $u \in -\mathcal{D}$ the corresponding rectangles (right triangles) are congruent to the ones corresponding to $-u \in \mathcal{D}$ by symmetry. In case (b), due to the periodicity of the functions involved in (4), the maximum and minimum are going to be given by critical points. We want to show that those critical points are in the range given in (c), as well.

Proof of Theorem 1.1. By Lagrange multipliers theorem (see [1, p. 135–137], Theorem 3.1) the critical points are given by the system of equations in u , v and λ ,

$$\left. \begin{array}{l} \frac{\partial f}{\partial u}(u, v) = \lambda \frac{\partial g}{\partial u}(u, v) \\ \frac{\partial f}{\partial v}(u, v) = \lambda \frac{\partial g}{\partial v}(u, v) \\ g(u, v) = 0 \end{array} \right\} \iff \left. \begin{array}{l} a \cos u - \cos(u + v) = \lambda (-a \sin u + \sin(u + v)) \\ b \cos v - \cos(u + v) = \lambda (-b \sin v + \sin(u + v)) \\ a \cos u + b \cos v - \cos(u + v) - ab = 0. \end{array} \right\} \quad (5)$$

One can check that $\frac{\partial f}{\partial u}(u, v) = \frac{\partial g}{\partial v}(u, v) = g(u, v) = 0$ leads to no solution in u and v if a and b are not 1. So, in order to apply Theorem 3.1 in [1, p. 136] we are going to work under this assumption and deal with this particular (simple) case separately.

Eliminating λ , we get the system:

$$\left. \begin{array}{l} -ab \cos u \sin v + a \cos u \sin(u + v) + b \cos(u + v) \sin v + ab \sin u \cos v \\ \quad - a \sin u \cos(u + v) - b \sin(u + v) \cos v = 0 \\ a \cos u + b \cos v - \cos(u + v) - ab = 0. \end{array} \right\} \quad (6)$$

Eliminating v from these equations, we get

$$\begin{aligned} \sin u (a^2 - 1)[a^2 b^4 - 4b^3 a^2 \cos u - 5b^2 a^2 \sin^2 u + 6a^2 b^2 + 2a^2 b \sin^2 u \cos u \\ - 4b a^2 \cos u + a^2 - a^2 \sin^2 u - b^4 \sin^2 u + 2b^3 \sin^2 u \cos u - b^2 \sin^2 u] = 0. \end{aligned} \quad (7)$$

We then get a family of critical points from the equation $\sin u = 0$. These turn out not to be relative extrema for the function f . Then other critical points may be given by the remaining factor of (7). After substituting $\sin^2 u = 1 - \cos^2 u$ and $\cos u = \omega$, the equation in question becomes

$$(-2\omega b + 1 + b^2)(a^2 b^2 - b^2 + b^2 \omega^2 - 2a^2 \omega b + a^2 \omega^2) = 0. \quad (8)$$

One solution of this equation is $\cos u = \bar{\omega} = \frac{b^2+1}{2b}$. Clearly $\bar{\omega} > 1$, by our assumption, and so this leads to no critical value of u . The other critical values are

$$\cos u_{1,2} = \frac{a^2 \pm b\sqrt{a^2 + b^2 - a^2 b^2}}{b^2 + a^2} = \frac{xy \pm zt}{x^2 + z^2}. \quad (9)$$

We are going to show that the global maximum, respectively, minimum are given by

$$\cos u_M = \frac{xy - zt}{x^2 + z^2}, \quad \text{respectively,} \quad \cos u_m = \frac{xy + zt}{x^2 + z^2}, \quad (10)$$

for a corresponding v_m, v_M determined obviously by u_m , respectively, u_M . Let us first show that these two values of $\cos u_M$ and $\cos u_m$ are indeed in the interval $[-1, 1]$.

Let $\alpha, \beta \in [0, \pi/2]$ be defined by $\cos \alpha = \frac{x}{\sqrt{x^2+z^2}}$, $\sin \alpha = \frac{z}{\sqrt{x^2+z^2}}$, $\cos \beta = \frac{y}{\sqrt{x^2+z^2}}$ and $\sin \beta = \frac{t}{\sqrt{x^2+z^2}}$. With this notation, the fractions in (10) become $\frac{xy-zt}{x^2+z^2} = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos(\alpha + \beta)$ and similarly, $\frac{xy+zt}{x^2+z^2} = \cos(\alpha - \beta)$.

Case I: $y \leq z$. Thus, (b) of Proposition 2.2 holds. In this case, if we construct $\triangle ABC$ with the positive orientation for every $u \in \mathcal{D}$, $\text{Area}(ABCD)$ in (2) as a function of u is continuous and periodic. Hence, the maximum and the minimum should appear as critical points. The values of u that give the maximum and minimum, respectively, are:

$$u_M = \alpha + \beta, \quad u_m = -|\alpha - \beta|. \quad (11)$$

Indeed, using the law of cosines in the triangle APB , we obtain $AB = \frac{xt+yz}{\sqrt{x^2+z^2}}$, and again the law of cosines for the side PB gives $2 \cdot AB \cdot x \cos(\angle PAB) = x^2 + AB^2 - y^2 = \frac{2x^2 t^2 + 2xyzt}{x^2+z^2} > 0$. This implies, in particular, that $m(\angle PAB) < \pi/2$. After more simplifications we obtain $\cos(\angle PAB) = \frac{t}{x^2+z^2} = \cos(\pi/2 - \beta)$, which means that $m(\angle PAB) = \pi/2 - \beta$. Hence $m(\angle PBA) = \pi/2 - \alpha$ and $m(\angle CBP) = \alpha$. Therefore, $\cos(\angle CBP) = \cos(\alpha) = \frac{x}{\sqrt{x^2+z^2}}$.

Now, we apply the law of cosines in $\triangle CBP$ to obtain $z^2 = y^2 + BC^2 - 2 \cdot BC \cdot y \cos(\angle CBP)$, or after completing the square $(BC - y \cos(\angle CBP))^2 = \left(\frac{zt}{\sqrt{x^2+z^2}}\right)^2$.

Because of reasons of orientation, and also because we want to maximize the area of $ABCD$, BC should be equal to $y \cos(\angle CBP) + \frac{zt}{\sqrt{x^2+z^2}} = \frac{xy+zt}{\sqrt{x^2+z^2}}$. This gives the area (still need to check it is maximal)

$$\text{Maxarea}(x, y, z) = AB \cdot BC = \frac{xt + yz}{\sqrt{x^2 + z^2}} \frac{xy + zt}{\sqrt{x^2 + z^2}} = xz + yt.$$

(For easy writing we use Maxarea for $\text{Maxarea}(x, y, z)$.)

Now, consider the case of $\sin u = 0$. That means that the point P is on the line determined by the side \overline{AB} , which implies that $AB = x + y$ (or, $AB = |x - y|$, when P is not on \overline{AB}). In any case, one obtains $BC = \sqrt{z^2 - y^2} = \sqrt{t^2 - x^2}$. Furthermore, the area becomes

$$AB \cdot BC = (x + y)\sqrt{z^2 - y^2} = x\sqrt{z^2 - y^2} + y\sqrt{t^2 - x^2} \leq xz + yt = \text{Maxarea}, \text{ or}$$

$$AB \cdot BC = |x - y|\sqrt{z^2 - y^2} \leq (x + y)\sqrt{z^2 - y^2} \leq xz + yt = \text{Maxarea}.$$

To show that Maxarea is indeed a global maximum we have to compare it with the other values of $\text{Area}(ABCD)$ obtained by choosing the other possible cases for u given by (9). Due to symmetry, the value of $\text{Area}(ABCD)$ (from (2)) for $u = -u_M$ can be computed as in the case $u = u_M$. One gets $\text{Area}(ABCD)(-u_M) = \frac{(xt+yz)(zt-xy)}{x^2+z^2} \leq \text{Maxarea}$. Then, we just need to compare Maxarea with the values of $\text{Area}(ABCD)$ at $u = \pm u_m$.

Now, if $u_m = -|\alpha - \beta|$, using the law of cosines in $\triangle APB$ we obtain $AB = \frac{|xt-yz|}{\sqrt{x^2+z^2}}$.

It is easy to see that $xt - yz = 0$ is equivalent to $y = x$. If $y = x$, then $y = z$ and therefore 0 is indeed the minimum area. So, we may assume that $y \neq x$. As before, similar calculations show that $2 \cdot AB \cdot y \cos(\angle PBA) = y^2 + AB^2 - x^2 = \frac{2z^2y^2 - 2xyzt}{x^2+z^2}$, or $|xt - yz| \cos(\angle PBA) = \frac{z(z y - x t)}{\sqrt{x^2+z^2}}$, which in turn gives $\cos(\angle PBA) = \pm \frac{z}{\sqrt{x^2+z^2}} = \pm \sin \alpha = \cos(\pi/2 \pm \alpha)$. This implies that $m(\angle PBA) = \pi/2 \pm \beta$. Analyzing the two different situations $y > x$ and $y < x$, we see from Fig. 4 that $m(\angle PBC) = \pi - \alpha$ if $y > x$, or $m(\angle PBC) = 2\pi - (\pi/2 + \alpha + \pi/2) = \pi - \alpha$ if $y < x$.

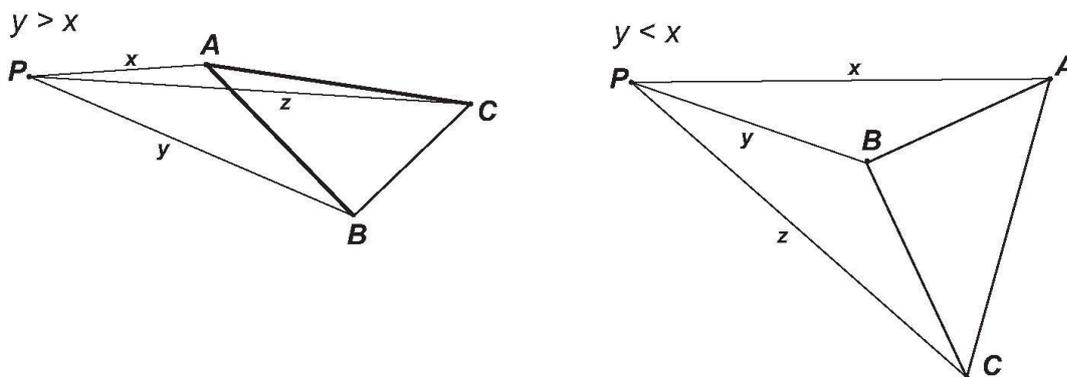


Fig. 4

So the measure of the angle $\angle PBC$ does not depend on the two cases $y < x$ or $y > x$. Hence, from $\triangle PBC$ we get $z^2 = y^2 + BC^2 - 2 \cdot BC \cdot y \cos(\angle PBC)$. But $\cos(\angle PBC) = -\cos \alpha = -\frac{x}{\sqrt{x^2+z^2}}$, and so after completing the square, BC satisfies the equation $(BC + xy/\sqrt{x^2+z^2})^2 = z^2 t^2/(x^2+z^2)$. This gives $BC = (zt - xy)/\sqrt{x^2+z^2}$. We have to point out that $zt > xy$ because this is equivalent to $z > y$, which is our assumption. Then

$$\text{Area}(ABCD) = AB \cdot BC = |xt - yz|(zt - xy)/(x^2 + z^2) = |xz - yt| := \text{Minarea}.$$

For $u = -u_m$ one obtains with a similar analysis $\text{Area}(ABCD)(-u_m) = |xt - yz|(zt + xy)/(x^2 + z^2)$. Since $\text{Area}(ABCD)(-u_m)$ and $\text{Area}(ABCD)(-u_M)$ are between Minarea and Maxarea, we have found the extreme values of the function $\text{Area}(ABCD)$ as a function of u .

Let us notice that we have solved in particular Problem 11057 of [3, p. 64], since in the case of maximum area, P is in the interior of the rectangle (all the angles $\angle PAB$, $\angle ABP$, $\angle PBC$ and $\angle PCB$ are acute angles, by similar calculations).

Now, we deal with the case of $a = 1$ and/or $b = 1$. It follows that $y = z$ and $x = t$. This means that P is on the perpendicular bisector of \overline{BC} and \overline{AD} determined in Fig. 5 by midpoints E and F .

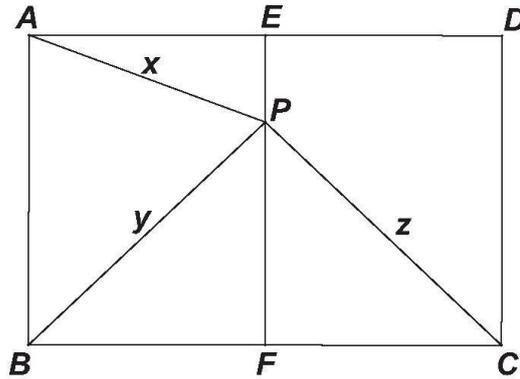


Fig. 5

Hence, the area of $ABCD$ is $2 \cdot \text{Area}(AEFB) = 2 \cdot 2 \cdot \text{Area}(AEB) = 4 \cdot \text{Area}(APB) = 4 \frac{xy \sin u}{2} = (xz + yt) \sin u \leq \text{Maxarea}$. In this situation, $\text{Minarea} = 0$.

Case II: $y > z$. So, $\mathcal{D} = [\arccos(\frac{y^2-z^2+tz}{xy}), \arccos(\frac{y^2-z^2-tz}{xy})] = [u_1, u_2]$.

In this case, as we mentioned before, we consider for each $u \in \mathcal{D}$ the two right triangles $\triangle ABC_1$ and $\triangle ABC_2$, which are both positively oriented and the area of the corresponding rectangles $f_1(u) := 2 \cdot \text{Area}(\triangle ABC_1)$ and $f_2(u) := 2 \cdot \text{Area}(\triangle ABC_2)$ ($BC_2 > BC_1$). We want to show that the maximum is attained on the branch f_2 and the minimum is attained on the branch f_1 . The difference between these two functions compared to (2) is that in f_1 the angle v is such that $\angle PC_2B$ is acute and in f_2 the angle v is such that $\angle PC_1B$ is obtuse.

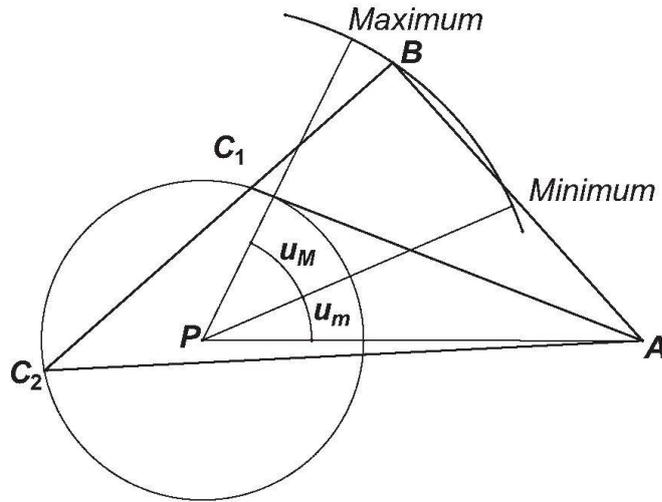


Fig. 6

In this case $u_M = \alpha + \beta$ and $u_m = |\alpha - \beta|$. Let us show that these angles are in \mathcal{D} . Since $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \frac{xy - zt}{x^2 + z^2}$, we need to show that

$$\frac{y^2 - z^2 - tz}{xy} \leq \frac{xy - zt}{x^2 + z^2} \leq \frac{y^2 - z^2 + tz}{xy}.$$

These two inequalities are in fact equivalent to the Cauchy-Schwartz inequality $(xy - zt)^2 \leq (x^2 + z^2)(y^2 + t^2)$.

Since f_1 [respectively f_2] agrees with (2) [for the appropriate choice of v] we obtain the same expressions for Maxarea and Minarea. At the endpoints of the interval \mathcal{D} , the two functions have the same values because $C_1 = C_2 := C$ ($v_1 = v_2$), and \overline{BC} becomes tangent to the circle $C(P, y)$. Moreover, $BC = \sqrt{y^2 - z^2}$ and $AB = t + z$ or $AB = |t - z|$. Hence, $f_2(u_1) = AB \cdot BC = \sqrt{y^2 - z^2}|t - z| \leq \sqrt{y^2 - z^2}(t + z) = f_2(u_2) = \sqrt{x^2 - t^2}z + \sqrt{y^2 - z^2}t \leq xz + yt = \text{Maxarea}$, and $f_1(u_2) = AB \cdot BC = \sqrt{y^2 - z^2}(t + z) \geq \sqrt{y^2 - z^2}|t - z| = f_1(u_1) = \sqrt{y^2 - z^2}|t - z| \geq |xz - yt| = \text{Minarea}$. The last inequality can be shown to be true by analyzing the two cases $y \leq x$ and $y > x$. The same analysis applies in this case to show that for the maximum area, the point P is in the interior of the rectangle and one can show that the minimum area is happening for a position of P in the exterior of the rectangle. \square

3 Further results and proposed problem

Obviously, the maximal area is an integer if x, y, z, t are integers. The following theorem provides a practical way to construct examples of such values of x, y, z (and t).

Theorem 3.1. *Let m, n, p, q be arbitrary positive integers. If x, y, z, t are given by $x = mn + pq$, $z = |mq - np|$, $y = |mn - pq|$ and $t = mq + np$, then Maxarea and Minarea are integers.*

Proof. Using [2, p. 15], we get that all integer solutions of the Diophantine equation $x^2 + z^2 = y^2 + t^2$ are of the mentioned forms. Since x, y, z, t are therefore integers, it follows that $\text{Maxarea}(x, y, z) = xz + yt$ is also an integer. Similarly, for Minarea . \square

Are these the only cases when Maxarea is an integer? Certainly not, since one can multiply the values of x, y, z and t given by Theorem 3.1 by \sqrt{l} (l positive integer which is not a perfect square), for instance, and still obtain an integer as a result.

Also, one may ask similar questions of finding the extreme values in various situations like having the circles in Problem 3 not necessarily concentric or on different curves. The following is a problem we propose for further study.

Problem 4. *Determine the extreme values for the area of a right triangle ABC ($\angle ABC = \pi/2$) with $A \in C(P, x)$, $B \in C(Q, y)$, $C \in C(R, z)$ where circle $C(Q, y)$ is contained in the interior of one of the other two circles.*

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