Zeitschrift: Elemente der Mathematik

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 61 (2006)

Artikel: Three points related to the incenter and excenters of a triangle

Autor: Odehnal, Boris

DOI: https://doi.org/10.5169/seals-1186

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 28.11.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Elemente der Mathematik

Three points related to the incenter and excenters of a triangle

Boris Odehnal

Boris Odehnal studierte von 1994 bis 1999 an der Technischen Universität Wien die Lehramtsfächer Mathematik und Darstellende Geometrie. Im Anschluß daran arbeitete er als Forschungsassistent am Institut für Geometrie der Technischen Universität Wien und gleichzeitig als Assistent an der Universität für Bodenkultur in Wien. Seit 2002 ist er am Institut für Diskrete Mathematik und Geometrie der Technischen Universität Wien tätig.

1 Introduction

Let $\Delta := \{A, B, C\}$ be a triangle in the Euclidean plane. The side lengths of Δ shall be denoted by $c := \overline{AB}$, $b := \overline{AC}$ and $a := \overline{BC}$. The interior angles enclosed by the edges of Δ are $\beta := \angle ABC$, $\gamma := \angle BCA$ and $\alpha := \angle CAB$, see Fig. 1.

It is well-known that the bisectors w_{α} , w_{β} and w_{γ} of the interior angles of Δ are concurrent in the *incenter I* of Δ . The bisectors $\overline{w_{\beta}}$ and $\overline{w_{\gamma}}$ of the exterior angles at the vertices A and B and w_{α} are concurrent in the center A_1 of the *excircle* touching Δ along BC from the outside.

Changing α , β and γ cyclically, we can find the remaining two *excenters* A_2 and A_3 . To get familiar with the notations used in this paper, see Fig. 1.

Here we remark that the base triangle Δ is the *orthoptic triangle* of the triangle built by the excenters. The orthocenter of Δ is the incenter of the orthoptic triangle. Later, when we give our theorems, a second interpretation will use this fact.

Die Inkreismitte und die drei Ankreismitten eines Dreiecks Δ sind die Mittelpunkte jener vier Kreise, die alle Seiten von Δ berühren. Aus diesen Punkten lassen sich drei weitere Punkte ableiten. Das so entstehende Dreieck Δ_S geht aus dem Dreieck Δ_A der Ankreismitten durch eine Halbdrehung um den FEUERBACH-Punkt von Δ hervor. Die Dreiecke Δ_A und Δ_S haben die EULER-Gerade und den Neunpunktekreis von FEUERBACH gemeinsam. In der vorliegenden Arbeit werden diese und weitere damit in Zusammenhang stehende Resultate mit elementaren Mitteln bewiesen.

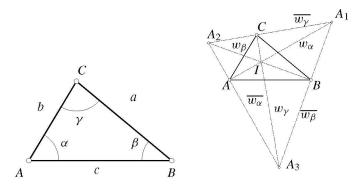


Fig. 1 Notations used in the paper

2 Results and theorems

Now we draw some normals emanating from the excenters A_i and the incenter I. We use the symbol $n_{A_1}(AC)$ to indicate that this line is perpendicular to AC and contains the point A_1 . Drawing the lines $n_{A_1}(AC)$, $n_{A_2}(BC)$ and $n_I(AB)$, respectively, we observe that these lines are concurrent in one point S_3 . Cyclic rearrangement of (A, B, C) and (1, 2, 3) enables us to state the following theorem:

Theorem 2.1 The following triples of lines are concurrent:

- (1) $(n_{A_1}(AC), n_{A_2}(BC), n_I(AB))$ are concurrent in S_3 .
- (2) $(n_{A_2}(AB), n_{A_3}(AC), n_I(BC))$ are concurrent in S_1 .
- (3) $(n_{A_3}(BC), n_{A_1}(AB), n_I(AC))$ are concurrent in S_2 .

Even in classical literature [3, 4] these points and the concurrencies of these normals are not mentioned. The concurrencies of the lines mentioned in Theorem 2.1 are illustrated in Fig. 2.

Moreover, we are able to prove the following result:

Theorem 2.2

- (1) The circumcenter of the triangle $\Delta_S := \{S_1, S_2, S_3\}$ is the incenter of Δ .
- (2) The circumradius of Δ_S equals twice the circumradius of Δ .

For the sake of simplicity we use the abbreviation $\Delta_A := \{A_1, A_2, A_3\}$ and state:

Theorem 2.3 The triangles Δ_A and Δ_S are congruent. There exists a rotation ρ about the center of the FEUERBACH circle of Δ_S with angle $\phi = \pi$ with $\rho(\Delta_A) = \Delta_S$.

Theorem 2.4

- (1) The FEUERBACH circle of Δ_S equals the circumcircle of Δ .
- (2) The triangles Δ_A and Δ_S share the FEUERBACH circle.

76 B. Odehnal

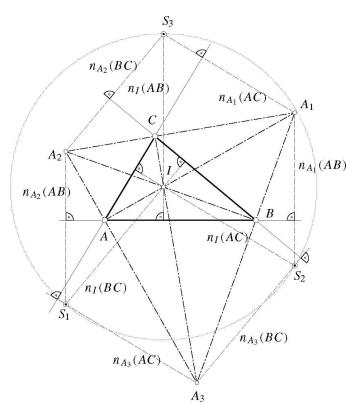


Fig. 2 Three remarkable points occuring as the intersection of some normals

Theorem 2.5

- (1) The incenter I lies on the EULER line e_{Δ_S} of Δ_S .
- (2) The triangles Δ_A and Δ_S share the EULER line.

None of the above theorems are hitherto known. Even in [3, 4] the points S_i and the theorems dealing with them are not mentioned.

3 Proof of the main results

Proof of Theorem 2.1. In order to show that the lines $n_{A_1}(AC)$, $n_{A_2}(BC)$ and $n_I(AB)$ are concurrent in S_3 , we compute the length of IS_3 in two different ways and obtain equal results. $\overline{IS_3}$ can be seen as the coordinate of the intersection points $n_I(AB) \wedge n_{A_1}(AC)$ and $n_I(AB) \wedge n_{A_2}(BC)$ on $n_I(AB)$.

We look at the triangles appearing in Fig. 3 and compute the length $\overline{IS_3}$. The first triangle to look at is $\Delta_1 := \{A, B, A_1\}$. The lengths of its edges are $\overline{AB} = c$, $\overline{BA_1}$ and $\overline{A_1A}$,

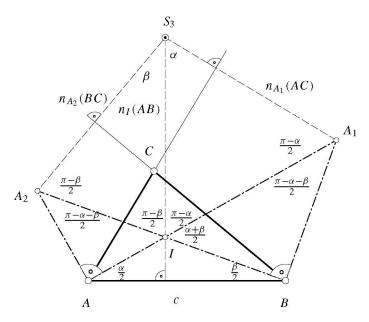


Fig. 3 Computation of $\overline{IS_3}$

respectively. The opposite angles have values $\frac{1}{2}(\pi - \alpha - \beta)$, $\alpha/2$ and $\frac{1}{2}(\pi + \beta)$. So we find

$$\overline{A_1B} = c \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha+\beta}{2}}.$$
 (1)

The next triangle we pay attention to is $\Delta_2 := \{B, I, A_1\}$. The lengths of edges appearing here are \overline{BI} , $\overline{IA_1}$ and $\overline{A_1B}$, respectively. The values of the angles lying opposite to these three edges are $\frac{1}{2}(\pi-\alpha-\beta)$, $\pi/2$ and $\frac{1}{2}(\alpha+\beta)$, respectively. Thus we find

$$\overline{IA_1} = 2c \frac{\sin \frac{\alpha}{2}}{\sin(\alpha + \beta)}.$$
 (2)

At last we look at $\Delta_3 := \{I, A_1, S_3\}$ with edge lengths $\overline{IA_1}$, $\overline{A_1S_3}$ and $\overline{S_3I}$, respectively. The angles opposite to these edges have values α , $\frac{1}{2}(\pi - \alpha)$ and $\frac{1}{2}(\pi - \alpha)$, respectively. Finally, we arrive at

$$\overline{IS_3} = \frac{c}{\sin(\alpha + \beta)} \,. \tag{3}$$

The computation of $\overline{IS_3}$ can be done in the same way with the triangles $\Delta_1':=\{A,B,A_2\}$, $\Delta_2':=\{A,I,A_2\}$ and $\Delta_3':=\{I,A_2,S_3\}$, which leads again to (3). Thus the coordinate of $n_I(AB) \wedge n_{A_1}(AC)$ and $n_I(AB) \wedge n_{A_2}(BC)$ on $n_I(AB)$ are equal and we have $n_I(AB) \wedge n_{A_1}(AC) = n_I(AB) \wedge n_{A_2}(BC) = S_3$.

78 B. Odehnal

Cyclic rearrangement of indices shows that the points S_1 and S_2 mentioned in Theorem 2.1 do exist and lie on the respective three normals.

We additionally obtain:

Theorem 3.1 The point S_i is the circumcenter of the triangle $\{I, A_j, A_k\}$ and (i, j, k) is either (1, 2, 3) or (2, 3, 1) or (3, 1, 2).

Proof. Looking at triangles $\Delta_4 := \{I, A_1, S_3\}$ and $\Delta_5 := \{I, A_2, S_3\}$ we find

$$\angle IA_1S_3 = \angle A_1IS_3$$
 and $\angle IA_2S_3 = \angle A_2IS_3$.

So we have $\overline{IS_3} = \overline{A_2S_3} = \overline{A_1S_3}$. Rearranging the indices completes the proof.

Proof of Theorem 2.2. Replacing (A, B, C) and (1, 2, 3) cyclically in (3) we obtain

$$\overline{IS_1} = \frac{a}{\sin(\beta + \gamma)}$$
 and $\overline{IS_2} = \frac{b}{\sin(\alpha + \gamma)}$. (4)

Since the values of the interior angles sum up to π , that is $\gamma = \pi - \alpha - \beta$, we find

$$\frac{c}{\sin(\alpha + \beta)} = \frac{c}{\sin \gamma} = 2R. \tag{5}$$

Further, we use the well-known formulae

$$\frac{c}{\sin \gamma} = \frac{b}{\sin \beta} = \frac{a}{\sin \alpha} = 2R, \tag{6}$$

which gives a simple relation between the angles, the side lengths and the circumradius R of Δ . Thus the circumradius of $\{S_1, S_2, S_3\}$ is twice the circumradius of Δ .

Proof of Theorem 2.3. In order to show that $\Delta_A \equiv \Delta_S$, we show that the lines A_1A_2 and S_1S_2 are parallel. (Equivalently, we could show that A_1A_3 and S_1S_3 are parallel and also A_3A_2 and S_3S_2 are parallel. Changing the indices while keeping the cycling ordering we obtain the equivalent results for the other pairs of lines.)

By definition we have $A_1A_2 \perp w_{\gamma}$ and from Theorem 2.2 we have $\overline{IS_1} = \overline{IS_2}$. Since w_{γ} is interior bisector of AC and BC it also is interior bisector of $n_I(AC)$ and $n_I(BC)$. Consequently, $S_1S_2 \perp w_{\gamma}$ and thus S_1S_2 is parallel to A_1A_2 .

Since $n_{A_2}(AB)$ and $n_{A_1}(AB)$ are parallel we have $\overline{A_1A_2} = \overline{S_1S_2}$. The same is true if we change indices (1, 2, 3) and (A, B, C), respectively, while keeping the cyclic ordering.

So far we have shown that Δ_A is congruent to Δ_S . Now we have to prove that there is a rotation ρ with angle π and $\rho(\Delta_A) = \Delta_S$.

We observe that A_2S_2 and A_3S_3 are the diagonals of the parallelogram $\Pi_1 := \{S_2, S_3, A_2, A_3\}$. Thus they intersect in a point X. Each of the parallelograms $\Pi_2 := \{S_1, S_3, A_1, A_3\}$ and $\Pi_3 := \{S_1, S_2, A_1, A_2\}$ shares a diagonal with Π_1 . Therefore, the diagonals of Π_1 ,

 Π_2 and Π_3 , respectively, are concurrent in X. Consequently, there exists a unique reflection about X which maps Δ_A to Δ_S . The existence of this reflection is equivalent to the existence of a rotation ρ about X with angle π transforming Δ_A into Δ_S .

At last we have to show that X is the FEUERBACH point F_{Δ_S} of Δ_S . The base triangle Δ is the pedal triangle of Δ_A . Thus the circumcircle of Δ is the FEUERBACH circle of Δ_A . Since ρ maps Δ_A to Δ_S it maps the corresponding pedal triangles onto each other by reflecting them about X. Thus the FEUERBACH circles of Δ_S and Δ_A coincide such as their centers coincide in X.

Proof of Theorem 2.4. There is nothing to be done. This theorem is a consequence of the proof of Theorem 2.3. \Box

Proof of Theorem 2.5. The incenter I of Δ is the circumcenter of Δ_S , see Theorem 2.1. Thus it is contained in the EULER line e_{Δ_S} of Δ_S .

Since e_{Δ_S} passes through the FEUERBACH point F_{Δ_S} , the rotation ρ with center F_{Δ_S} transforms e_{Δ_S} into e_{Δ_A} .

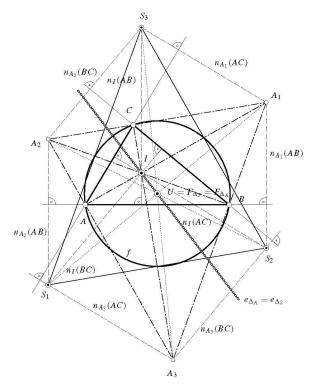


Fig. 4 Triangles Δ_A and Δ_S with common EULER line $e_{\Delta_A}=e_{\Delta_S}$ and FEUERBACH circle f

80 B. Odehnal

4 Alternative interpretation

As remarked in Section 1, the theorems given in Section 2 can be seen in a different light. For a given triangle $\Delta_A := \{A_1, A_2, A_3\}$ draw the orthoptic triangle $\Delta_O := \{B_1, B_2, B_3\}$, where $B_i \in A_j A_k$ with cyclic ordering of (i, j, k). The orthocenter H of Δ_A is the incenter of Δ_O . Now we recall that Δ_A is the excenter triangle of Δ_O .

Thus Theorem 2.1 can be reformulated:

Theorem 4.1 (Equivalent to Theorem 2.1)

The normals from the vertex A_1 of the base triangle Δ_A to the side B_1B_2 of the orthoptic triangle Δ_O , the normal from A_3 to B_2B_3 , and the normal through the orthocenter H of Δ_A to B_1B_3 are concurrent in a point S_2 .

This remains true if we change the indices while keeping the cyclic ordering. \Box

Acknowledgements. I have to express my sincere thanks to the reviewer for his hints and comments.

References

- [1] Coxeter, H.S.M.: Introduction to Geometry. J. Wiley & Sons, New York-London 1961.
- [2] Kimberling, C.: Encyclopedia of Triangle Centers. Available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html
- [3] Schwarz, H.A.: Gesammelte mathematische Abhandlungen. Springer, Berlin 1890.
- [4] Steiner, J.: Jacob Steiner's Gesammelte Werke. (2 Bände) Reimer, Berlin 1881.

Boris Odehnal Institut für Geometrie Technische Universität Wien Wiedner Hauptstr. 8–10/113 A–1040 Wien, Österreich

e-mail: boris@geometrie.tuwien.ac.at