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Elemente der Mathematik

On a conjecture about relative lengths

Zhanjun Su and Ren Ding

Zhanjun Su received his B.Sc., M.Sc., and his Ph.D. from Hebei Normal University in Shijiazhuang (China), where he is now associate professor of mathematics. His research interests focus primarily on discrete, convex, and combinatorial geometry.

Ren Ding is professor of mathematics and supervisor of the Ph.D. programs at Hebei Normal University in Shijiazhuang (China). His research interests focus primarily on discrete, convex, and combinatorial geometry.

We need some definitions from [1]. Let $\mathcal{C} \subset \mathbb{R}^2$ be a convex body. A chord pq of $\mathcal{C}$ is called an affine diameter of $\mathcal{C}$, if there is no longer parallel chord in $\mathcal{C}$. The ratio of $|ab|$ to $\frac{1}{2}|a'b'|$, where $a'b'$ is an affine diameter of $\mathcal{C}$ parallel to ab, is called the *$\mathcal{C}$-length* of ab, or the *relative length* of ab, if there is no doubt about $\mathcal{C}$. We denote it by $\lambda_{\mathcal{C}}(ab)$.

Denote by λ_n the relative length of a side of the regular n-gon. For every $ab \subset \mathcal{C}$ we have $|ab| \leq |a'b'|$, where $a'b'$ is the affine diameter parallel to ab, hence $0 < \lambda_n = \frac{|ab|}{|a'b'|/2} \leq 2$. For every regular triangle (or square), since its side length equals its corresponding affine diameter, $\lambda_3 = \lambda_4 = 2$. Let $\mathcal{C} = abcde$ be a regular pentagon with side length 1, join the points c and e, then we know that ab is parallel to ce and $\lambda_5 = \lambda_{\mathcal{C}}(ab) = \frac{|ab|}{|ce|/2} = 1/\cos(\frac{\pi}{5}) = \sqrt{5} - 1$ (see Fig. 1). Let $\mathcal{C} = abcdef$ be a regular hexagon with side length 1, join the points c and f, then ab is parallel to cf and $\lambda_6 = \frac{|ab|}{|cf|/2} = 1$ (see Fig. 2).

A side ab of a convex n-gon $\mathcal{P}$ is called *relatively short* if $\lambda_{\mathcal{P}}(ab) \leq \lambda_n$, and it is called *relatively long* if $\lambda_{\mathcal{P}}(ab) \geq \lambda_n$.

Sind $\mathcal{C}$ eine konvexe Figur und ab eine Strecke der Euklidischen Ebene, so wird im nachfolgenden Beitrag das Verhältnis der Länge $|ab|$ zur Hälfte der Länge einer längsten Sehne von $\mathcal{C}$ untersucht, die parallel zu ab ist. Dieses Verhältnis wird relative Länge von ab genannt und mit $\lambda_{\mathcal{C}}(ab)$ bezeichnet; die relative Länge einer Seite eines regelmässigen n-Ecks wird durch λ_n abgekürzt. Beispielsweise gilt $\lambda_3 = \lambda_4 = 2$, $\lambda_5 = \sqrt{5} - 1$ und $\lambda_6 = 1$. Unter anderem bestätigen die Autoren im folgenden eine Vermutung von K. Doliwka und M. Lassak, welche besagt, dass jedes konvexe Sechseck eine Seite der relativen Länge kleiner oder gleich $8 - 4\sqrt{3} = 1,071\ldots$ besitzt.

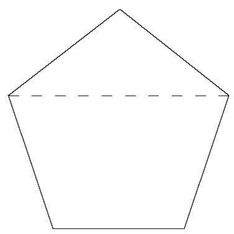

Fig. 1

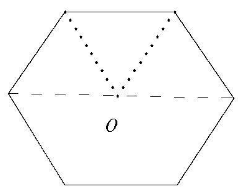

Fig. 2

In [1] Doliwka and Lassak showed that every convex pentagon (or quadrangle) has a relatively short side and a relatively long side. They conjectured that *every convex hexagon has a side of relative length at most* $8 - 4\sqrt{3} = 1.071\ldots$. We prove that this is true.

First, we give a hexagon which does not have any relatively short side. Let $\mathcal{H} = abcdef$ be a hexagon, where $\triangle bdf$ is a regular triangle, $|ab| = |bc| = |cd| = |de| = |ef| = |fa| = 1$, and $|ad| = |be| = |cf| = |bd|$ (see Fig. 3). It is easy to show that $ab \perp bc$. Draw $fm \perp bc$. Obviously, $|a'b'| = |fm| = \frac{1}{2}\tan(\frac{5\pi}{12}) = \frac{2+\sqrt{3}}{2}$, and we obtain $\lambda_{\mathcal{H}}(ab) = \frac{4}{2+\sqrt{3}} = 8 - 4\sqrt{3}$. In this way, we obtain that $\lambda_{\mathcal{H}}(ab) = \lambda_{\mathcal{H}}(bc) = \lambda_{\mathcal{H}}(cd) = \lambda_{\mathcal{H}}(de) = \lambda_{\mathcal{H}}(ef) = \lambda_{\mathcal{H}}(fa) = 8 - 4\sqrt{3} = 1.071\ldots > 1 = \lambda_6$. So, as a matter of fact, each side of the hexagon is relatively long.

Theorem 1. *Every convex hexagon has a side of relative length at most* $8 - 4\sqrt{3} = 1.071\ldots$, *and this upper bound is tight.*

Let $\mathcal{H}$ be a convex hexagon with vertices a, c', b, a', c, b'. For every non-degenerate affine transformation τ and for arbitrary points $p, q \in \mathcal{C}$, we know that $\lambda_{\mathcal{C}}(pq) = \lambda_{\tau(\mathcal{C})}(\tau(p)\tau(q))$. Thus, without loss of generality, we may assume that three non-adjacent vertices of the convex hexagon $\mathcal{H}$ form a regular triangle $\triangle abc$.

Let the center of $\triangle abc$ be o, and denote by $\overline{ao}$ the straight line passing through a, o. Similarly, we define straight lines $\overline{bo}$, $\overline{co}$. A convex hexagon $\mathcal{H} = ac'ba'cb'$ is called

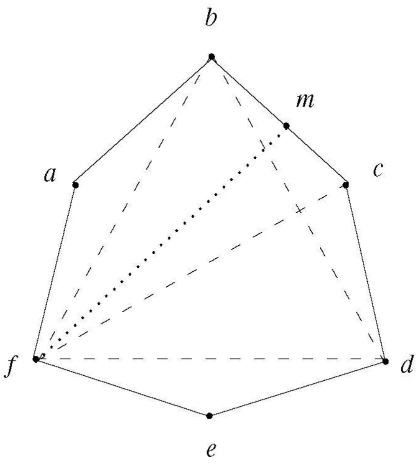

Fig. 3

a *special-regular hexagon*, if $\triangle abc$ is a regular triangle and $|aa'| = |bb'| = |cc'|$ with $a' \in \overline{ao}$, $b' \in \overline{bo}$, $c' \in \overline{co}$ (see Fig. 4).

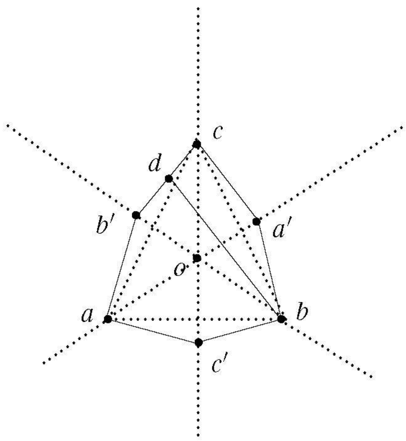

Fig. 4

Lemma 1. *The relative length of each side of a special-regular hexagon* $\mathcal{H} = ac'ba'cb'$ *is at most* $8 - 4\sqrt{3}$.

Proof. Without loss of generality, let $a = (-1, 0)$, $b = (1, 0)$, $c = (0, \sqrt{3})$, and $|aa'| = t > \sqrt{3}$. Then,

$$a' = \left(\frac{\sqrt{3}t}{2} - 1, \frac{t}{2}\right), \quad b' = \left(1 - \frac{\sqrt{3}t}{2}, \frac{t}{2}\right).$$

Take a point $d \in cb'$ such that the segments $a'c$ and bd are parallel (see Fig. 4). We then easily compute

$$d = \Big(\frac{2t - 2\sqrt{3}}{t - 2\sqrt{3}}, \frac{-t}{2 - \sqrt{3}t}\Big),$$

which leads to

$$|a'c|^2 = t^2 - 2\sqrt{3}t + 4, \quad |bd|^2 = t^2\Big(\frac{1}{(t - 2\sqrt{3})^2} + \frac{1}{(2 - \sqrt{3}t)^2}\Big).$$

Hence, we find

$$\lambda_{\mathcal{H}}(a'c) = \frac{2|a'c|}{|bd|} = \frac{-\sqrt{3}t^2 + 8t - 4\sqrt{3}}{t} = 8 - \sqrt{3}\Big(t + \frac{4}{t}\Big) \le 8 - 4\sqrt{3}.$$

Similarly, we can compute the relative length for each side of the hexagon and Lemma 1 is proved. □

Remark 1. When $t = 2$, that is, $|aa'| = |ab|$, we get the hexagon in Fig. 3, and the upper bound $8 - 4\sqrt{3}$ is attained. Generally speaking, when $\sqrt{3} < t < \frac{4\sqrt{3}}{3}$, we have $8 - \sqrt{3}(t + \frac{4}{t}) > 1$, so when $\sqrt{3} < t < \frac{4\sqrt{3}}{3}$, each side of the hexagon is relatively long.

Lemma 2. *If $\triangle abc$ is a regular triangle, $a' \in \overline{ao}$, $b' \in \overline{bo}$, $c' \in \overline{co}$ (see Fig. 5), then the convex hexagon $\mathcal{H} = ac'ba'cb'$ has a side of relative length at most* $8 - 4\sqrt{3}$.

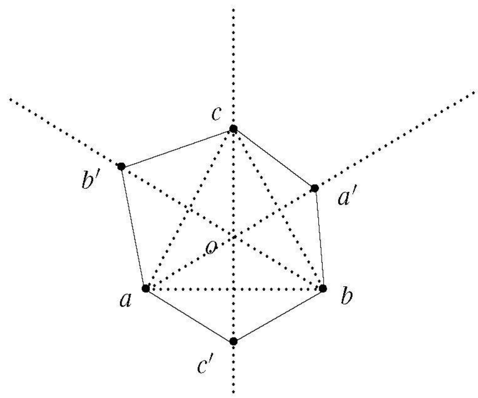

Fig. 5

Proof. Consider the segments aa', bb', and cc'. If $|aa'| = |bb'| = |cc'|$, then $\mathcal{H}$ is a special-regular hexagon, and we reach the conclusion by Lemma 1. Otherwise, we may assume that $|aa'| = \min\{|aa'|, |bb'|, |cc'|\}$. Then there exist points $b'' \in bb'$ and $c'' \in cc'$ such that $|aa'| = |bb''| = |cc''|$, and hence $\mathcal{H}_1 = ac''ba'cb''$ is a special-regular hexagon contained in hexagon $\mathcal{H} = ac'ba'cb'$. Therefore, $\lambda_{\mathcal{H}}(a'c) \le \lambda_{\mathcal{H}_1}(a'c) = 8 - 4\sqrt{3}$. Lemma 2 is proved. □

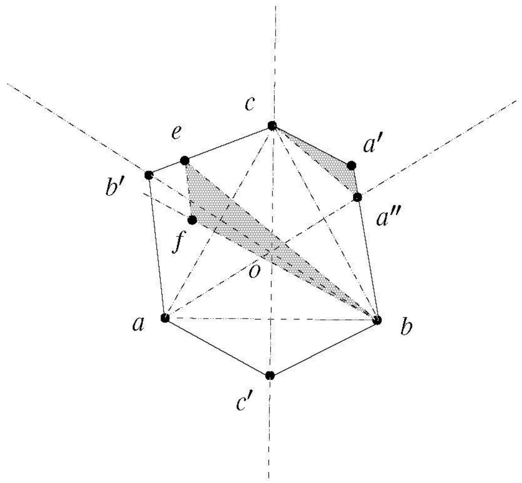

Fig. 6

Lemma 3. *If $\triangle abc$ is a regular triangle, $a' \notin \overline{ao}$, $b' \in \overline{bo}$, $c' \in \overline{co}$ (see Fig. 6), then the convex hexagon $\mathcal{H} = ac'ba'cb'$ has a side of relative length at most* $8 - 4\sqrt{3}$.

Proof. Denote by a'' the intersecting point of the segments ba' and $\overline{ao}$. Consider the convex hexagon $\mathcal{H}_1 = ac'ba''cb'$. Obviously, the hexagon $\mathcal{H}_1 \subseteq \mathcal{H} = ac'ba'cb'$. If $|bb'|$ or $|cc'|$ equals $\min\{|aa''|, |bb'|, |cc'|\}$, then, according to Lemma 2, $\lambda_{\mathcal{H}}(ab')$ or $\lambda_{\mathcal{H}}(ac')$ is at most $8 - 4\sqrt{3}$. If $|aa''|$ equals $\min\{|aa''|, |bb'|, |cc'|\}$, then, by Lemma 2, $|ca''|$ is at most $8 - 4\sqrt{3}$. Without loss of generality, we may assume that $|aa'| = |bb'| = |aa''|$, see Fig. 6. Then there exist points e and f such that $be \parallel a''c$, $bf \parallel a'c$, and $ef \parallel a'a''$. Then, $\triangle ca'a'' \sim \triangle bfe$, hence, $\frac{2|a''c|}{|be|} = \frac{2|a'c|}{|bf|} \le 8 - 4\sqrt{3}$. Since $|bf|$ is smaller than the affine diameter parallel to $a'c$, therefore, $\lambda_{\mathcal{H}}(a'c) \le 8 - 4\sqrt{3}$. The proof is complete. □

Remark 2. For the case $a' \in \overline{ao}$, $b' \notin \overline{bo}$, $c' \in \overline{co}$, or the case $a' \in \overline{ao}$, $b' \in \overline{bo}$, $c' \notin \overline{co}$ the conclusion of Lemma 3 can be reached similarly.

Lemma 4. *If $\triangle abc$ is a regular triangle, $a' \notin \overline{ao}$, $b' \notin \overline{bo}$, $c' \in \overline{co}$ (see Fig. 7), then the convex hexagon $\mathcal{H} = ac'ba'cb'$ has a side of relative length at most* $8 - 4\sqrt{3}$.

Proof. Denote by a'' the intersecting point of ba' and $\overline{ao}$, and b'' the intersecting point of cb' and $\overline{bo}$. If the hexagon $\mathcal{H}_1 = ac'ba''cb''$ is a special-regular hexagon, then the points a and c' are distant in relative length by at most $8 - 4\sqrt{3}$. Otherwise, we have three cases to consider. When $|cc'| = \min\{|aa''|, |bb''|, |cc'|\}$, by Lemma 2 we obtain $\lambda_{\mathcal{H}}(ac') \le 8 - 4\sqrt{3}$; when $|aa''| = \min\{|aa''|, |bb''|, |cc'|\}$, then by Lemma 2 we have $\lambda_{\mathcal{H}_1}(ca'') \le 8 - 4\sqrt{3}$, and so $\lambda_{\mathcal{H}}(ca') \le 8 - 4\sqrt{3}$; when $|bb''| = \min\{|aa''|, |bb''|, |cc'|\}$, then by Lemma 2 we have $\lambda_{\mathcal{H}_1}(ab'') \le 8 - 4\sqrt{3}$, and so $\lambda_{\mathcal{H}}(ab') \le 8 - 4\sqrt{3}$. The proof is complete. □

Remark 3. For the case $a' \notin \overline{ao}$, $b' \in \overline{bo}$, $c' \notin \overline{co}$, or the case $a' \in \overline{ao}$, $b' \notin \overline{bo}$, $c' \notin \overline{co}$ the conclusion of Lemma 4 can be reached similarly.

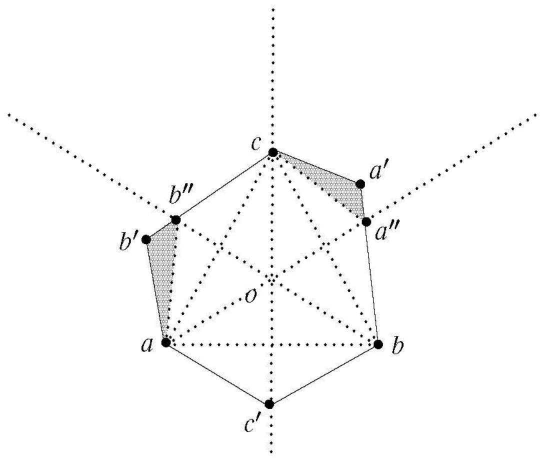

Fig. 7

Lemma 5. *If $\triangle abc$ is a regular triangle, $a' \notin \overline{ao}$, $b' \notin \overline{bo}$, $c' \notin \overline{co}$ (see Fig. 8), then the convex hexagon $\mathcal{H} = ac'ba'cb'$ has a side of relative length at most* $8 - 4\sqrt{3}$.

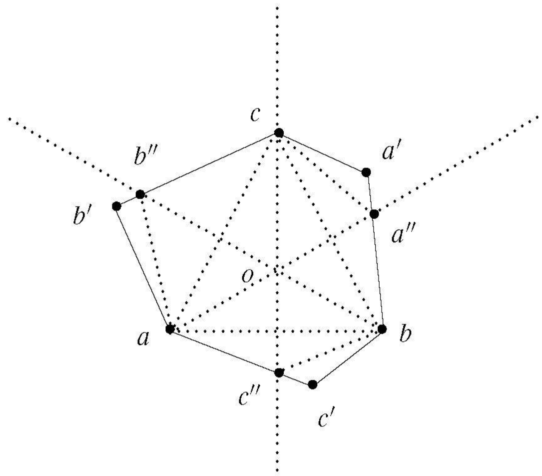

Fig. 8

Proof. Denote by a'' the intersecting point of ba' and $\overline{ao}$, b'' the intersecting point of cb' and $\overline{bo}$, and c'' the intersecting point of ac' and $\overline{co}$. Consider the hexagon $\mathcal{H}_1 = ac''ba''cb''$. Without loss of generality we may assume that $|aa''| = \min\{|aa''|, |bb''|, |cc''|\}$, then by Lemma 2 we have $\lambda_{\mathcal{H}_1}(ca'') \le 8 - 4\sqrt{3}$ and hence $\lambda_{\mathcal{H}}(ca') \le 8 - 4\sqrt{3}$. The proof is complete. $\square$

Proof of Theorem 1. Combining Lemmas 1–5 we obtain Theorem 1, that is, the conjecture of Doliwka and Lassak is true. By Remark 1 the upper bound is tight. □

From Remark 1 we know that many convex hexagons have relative long sides, but not every convex polygon has a relative long side. For example we have the following result:

Theorem 2. *There exists a convex* 12*-gon which does not have any relatively long side.*

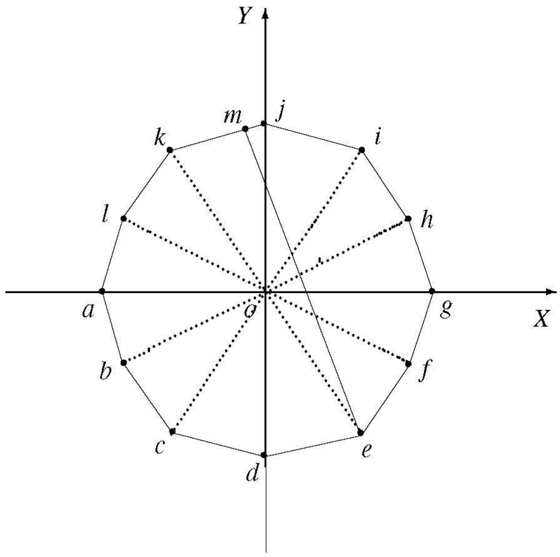

Fig. 9

Proof. We consider the convex 12-gon $Q = abcdefghijkl$, as shown in Fig. 9, where $|oa| = |oc| = |oe| = |og| = |oi| = |ok| = t$ with $1 \leq t < \frac{2\sqrt{3}}{3}$, $|ob| = |od| = |of| = |oh| = |oj| = |ol| = 1$, the angle formed by any two consecutive segments with common point o equals $\frac{\pi}{6}$. By the symmetry of the construction of the convex 12-gon all sides of the 12-gon have the same relative length. We need only to compute one of them, say, $\lambda_Q(gh)$. Obviously, $e = (\frac{t}{2}, -\frac{\sqrt{3}t}{2})$, $g = (t, 0)$, $h = (\frac{\sqrt{3}}{2}, \frac{1}{2})$, $j = (0, 1)$, $k = (-\frac{t}{2}, \frac{\sqrt{3}t}{2})$, and there exists a point $m \in jk$ such that the segments gh and em are parallel (see Fig. 9). The computation shows

$$|gh| = \sqrt{t^2 - \sqrt{3}t + 1}, \quad m = \Big(\frac{\sqrt{3}t^3 - \sqrt{3}t}{2\sqrt{3}t^2 - 8t + 2\sqrt{3}}, \frac{-3t^3 + 4\sqrt{3}t^2 - 5t}{2\sqrt{3}t^2 - 8t + 2\sqrt{3}}\Big),$$

hence,

$$|em| = \frac{2t\sqrt{t^2 - \sqrt{3}t + 1}}{|\sqrt{3}t^2 - 4t + \sqrt{3}|}.$$

Therefore, we obtain

$$\lambda_Q(gh) = \frac{2|gh|}{|em|} = \frac{-\sqrt{3}t^2 + 4t - \sqrt{3}}{t} = 4 - \sqrt{3}\Big(t + \frac{1}{t}\Big) \leq 2(2 - \sqrt{3}).$$

However, we can easily obtain that $\lambda_{12} = 2(2 - \sqrt{3})$ by setting $t = 1$ in the above equation since Q is a regular 12-gon when $t = 1$. Therefore, $\lambda_Q(gh) \leq \lambda_{12}$ and the proof is complete. □

Acknowledgement

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References

[1] Doliwka, K.; Lassak, M.: On Relatively Short and Long Sides of Convex Pentagons. *Geom. Dedicata* 56 (1995), 221–224.

Zhanjun Su and Ren Ding
Mathematics Department
Hebei Normal University
Yuhua Road
Shijiazhuang 050016
People's Republic of China
e-mail: suzj888@163.com
rending@heinfo.net