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## On a conjecture about relative lengths

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We need some definitions from [1]. Let  $\mathcal{C} \subset \mathbb{R}^2$  be a convex body. A chord  $pq$  of  $\mathcal{C}$  is called an affine diameter of  $\mathcal{C}$ , if there is no longer parallel chord in  $\mathcal{C}$ . The ratio of  $|ab|$  to  $\frac{1}{2}|a'b'|$ , where  $a'b'$  is an affine diameter of  $\mathcal{C}$  parallel to  $ab$ , is called the  $\mathcal{C}$ -length of  $ab$ , or the relative length of  $ab$ , if there is no doubt about  $\mathcal{C}$ . We denote it by  $\lambda_{\mathcal{C}}(ab)$ .

Denote by  $\lambda_n$  the relative length of a side of the regular  $n$ -gon. For every  $ab \subset \mathcal{C}$  we have  $|ab| \leq |a'b'|$ , where  $a'b'$  is the affine diameter parallel to  $ab$ , hence  $0 < \lambda_n = \frac{|ab|}{|a'b'|/2} \leq 2$ . For every regular triangle (or square), since its side length equals its corresponding affine diameter,  $\lambda_3 = \lambda_4 = 2$ . Let  $\mathcal{C} = abcde$  be a regular pentagon with side length 1, join the points  $c$  and  $e$ , then we know that  $ab$  is parallel to  $ce$  and  $\lambda_5 = \lambda_{\mathcal{C}}(ab) = \frac{|ab|}{|ce|/2} = 1/\cos(\frac{\pi}{5}) = \sqrt{5} - 1$  (see Fig. 1). Let  $\mathcal{C} = abcdef$  be a regular hexagon with side length 1, join the points  $c$  and  $f$ , then  $ab$  is parallel to  $cf$  and  $\lambda_6 = \frac{|ab|}{|cf|/2} = 1$  (see Fig. 2).

A side  $ab$  of a convex  $n$ -gon  $\mathcal{P}$  is called *relatively short* if  $\lambda_{\mathcal{P}}(ab) \leq \lambda_n$ , and it is called *relatively long* if  $\lambda_{\mathcal{P}}(ab) \geq \lambda_n$ .

Sind  $\mathcal{C}$  eine konvexe Figur und  $ab$  eine Strecke der Euklidischen Ebene, so wird im nachfolgenden Beitrag das Verhältnis der Länge  $|ab|$  zur Hälfte der Länge einer längsten Sehne von  $\mathcal{C}$  untersucht, die parallel zu  $ab$  ist. Dieses Verhältnis wird relative Länge von  $ab$  genannt und mit  $\lambda_{\mathcal{C}}(ab)$  bezeichnet; die relative Länge einer Seite eines regelmässigen  $n$ -Ecks wird durch  $\lambda_n$  abgekürzt. Beispielsweise gilt  $\lambda_3 = \lambda_4 = 2$ ,  $\lambda_5 = \sqrt{5} - 1$  und  $\lambda_6 = 1$ . Unter anderem bestätigen die Autoren im folgenden eine Vermutung von K. Doliwka und M. Lassak, welche besagt, dass jedes konvexe Sechseck eine Seite der relativen Länge kleiner oder gleich  $8 - 4\sqrt{3} = 1,071\dots$  besitzt.

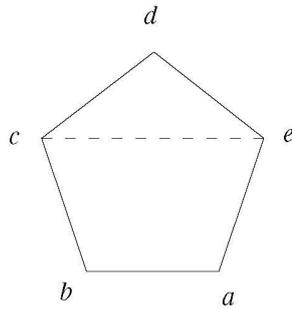


Fig. 1

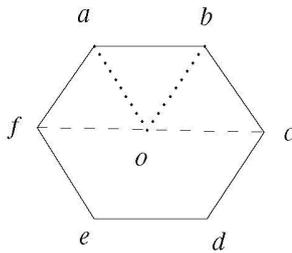


Fig. 2

In [1] Doliwka and Lassak showed that every convex pentagon (or quadrangle) has a relatively short side and a relatively long side. They conjectured that *every convex hexagon has a side of relative length at most  $8 - 4\sqrt{3} = 1.071\dots$* . We prove that this is true.

First, we give a hexagon which does not have any relatively short side. Let  $\mathcal{H} = abcdef$  be a hexagon, where  $\triangle bdf$  is a regular triangle,  $|ab| = |bc| = |cd| = |de| = |ef| = |fa| = 1$ , and  $|ad| = |be| = |cf| = |bd|$  (see Fig. 3). It is easy to show that  $ab \perp bc$ . Draw  $fm \perp bc$ . Obviously,  $|a'b'| = |fm| = \frac{1}{2} \tan(\frac{5\pi}{12}) = \frac{2+\sqrt{3}}{2}$ , and we obtain  $\lambda_{\mathcal{H}}(ab) = \frac{4}{2+\sqrt{3}} = 8 - 4\sqrt{3}$ . In this way, we obtain that  $\lambda_{\mathcal{H}}(ab) = \lambda_{\mathcal{H}}(bc) = \lambda_{\mathcal{H}}(cd) = \lambda_{\mathcal{H}}(de) = \lambda_{\mathcal{H}}(ef) = \lambda_{\mathcal{H}}(fa) = 8 - 4\sqrt{3} = 1.071\dots > 1 = \lambda_6$ . So, as a matter of fact, each side of the hexagon is relatively long.

**Theorem 1.** *Every convex hexagon has a side of relative length at most  $8 - 4\sqrt{3} = 1.071\dots$ , and this upper bound is tight.*

Let  $\mathcal{H}$  be a convex hexagon with vertices  $a, c', b, a', c, b'$ . For every non-degenerate affine transformation  $\tau$  and for arbitrary points  $p, q \in \mathcal{C}$ , we know that  $\lambda_{\mathcal{C}}(pq) = \lambda_{\tau(\mathcal{C})}(\tau(p)\tau(q))$ . Thus, without loss of generality, we may assume that three non-adjacent vertices of the convex hexagon  $\mathcal{H}$  form a regular triangle  $\triangle abc$ .

Let the center of  $\triangle abc$  be  $o$ , and denote by  $\overline{ao}$  the straight line passing through  $a, o$ . Similarly, we define straight lines  $\overline{bo}, \overline{co}$ . A convex hexagon  $\mathcal{H} = ac'ba'cb'$  is called

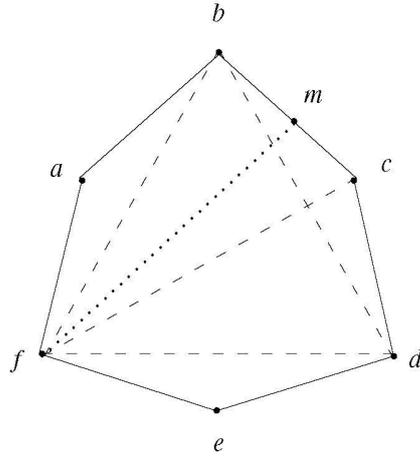


Fig. 3

a *special-regular hexagon*, if  $\triangle abc$  is a regular triangle and  $|aa'| = |bb'| = |cc'|$  with  $a' \in \overline{ao}$ ,  $b' \in \overline{bo}$ ,  $c' \in \overline{co}$  (see Fig. 4).

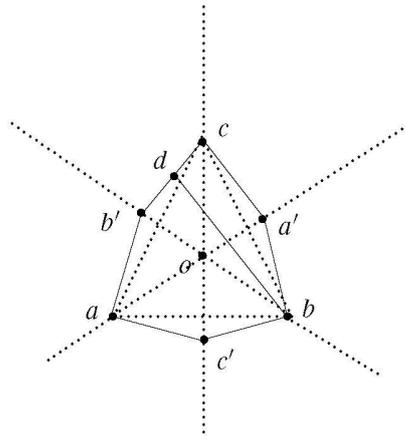


Fig. 4

**Lemma 1.** *The relative length of each side of a special-regular hexagon  $\mathcal{H} = ac'ba'cb'$  is at most  $8 - 4\sqrt{3}$ .*

*Proof.* Without loss of generality, let  $a = (-1, 0)$ ,  $b = (1, 0)$ ,  $c = (0, \sqrt{3})$ , and  $|aa'| = t > \sqrt{3}$ . Then,

$$a' = \left( \frac{\sqrt{3}t}{2} - 1, \frac{t}{2} \right), \quad b' = \left( 1 - \frac{\sqrt{3}t}{2}, \frac{t}{2} \right).$$

Take a point  $d \in cb'$  such that the segments  $a'c$  and  $bd$  are parallel (see Fig. 4). We then easily compute

$$d = \left( \frac{2t - 2\sqrt{3}}{t - 2\sqrt{3}}, \frac{-t}{2 - \sqrt{3}t} \right),$$

which leads to

$$|a'c|^2 = t^2 - 2\sqrt{3}t + 4, \quad |bd|^2 = t^2 \left( \frac{1}{(t - 2\sqrt{3})^2} + \frac{1}{(2 - \sqrt{3}t)^2} \right).$$

Hence, we find

$$\lambda_{\mathcal{H}}(a'c) = \frac{2|a'c|}{|bd|} = \frac{-\sqrt{3}t^2 + 8t - 4\sqrt{3}}{t} = 8 - \sqrt{3} \left( t + \frac{4}{t} \right) \leq 8 - 4\sqrt{3}.$$

Similarly, we can compute the relative length for each side of the hexagon and Lemma 1 is proved.  $\square$

**Remark 1.** When  $t = 2$ , that is,  $|aa'| = |ab|$ , we get the hexagon in Fig. 3, and the upper bound  $8 - 4\sqrt{3}$  is attained. Generally speaking, when  $\sqrt{3} < t < \frac{4\sqrt{3}}{3}$ , we have  $8 - \sqrt{3} \left( t + \frac{4}{t} \right) > 1$ , so when  $\sqrt{3} < t < \frac{4\sqrt{3}}{3}$ , each side of the hexagon is relatively long.

**Lemma 2.** If  $\triangle abc$  is a regular triangle,  $a' \in \overline{ao}$ ,  $b' \in \overline{bo}$ ,  $c' \in \overline{co}$  (see Fig. 5), then the convex hexagon  $\mathcal{H} = ac'ba'cb'$  has a side of relative length at most  $8 - 4\sqrt{3}$ .

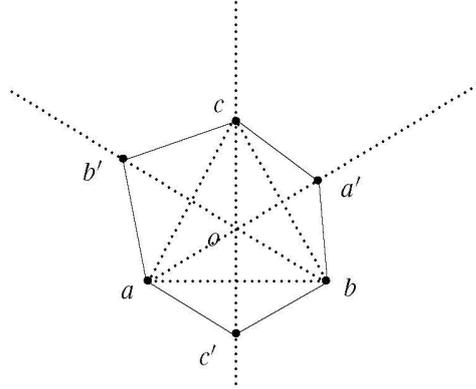


Fig. 5

*Proof.* Consider the segments  $aa'$ ,  $bb'$ , and  $cc'$ . If  $|aa'| = |bb'| = |cc'|$ , then  $\mathcal{H}$  is a special-regular hexagon, and we reach the conclusion by Lemma 1. Otherwise, we may assume that  $|aa'| = \min\{|aa'|, |bb'|, |cc'|\}$ . Then there exist points  $b'' \in bb'$  and  $c'' \in cc'$  such that  $|aa'| = |bb''| = |cc''|$ , and hence  $\mathcal{H}_1 = ac''ba'cb''$  is a special-regular hexagon contained in hexagon  $\mathcal{H} = ac'ba'cb'$ . Therefore,  $\lambda_{\mathcal{H}}(a'c) \leq \lambda_{\mathcal{H}_1}(a'c) = 8 - 4\sqrt{3}$ . Lemma 2 is proved.  $\square$

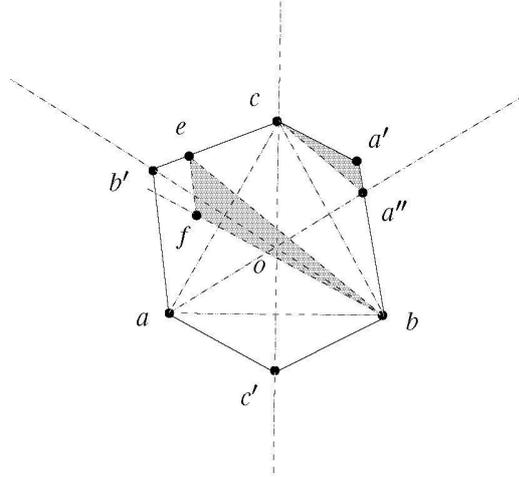


Fig. 6

**Lemma 3.** *If  $\triangle abc$  is a regular triangle,  $a' \notin \overline{ao}$ ,  $b' \in \overline{bo}$ ,  $c' \in \overline{co}$  (see Fig. 6), then the convex hexagon  $\mathcal{H} = ac'ba'cb'$  has a side of relative length at most  $8 - 4\sqrt{3}$ .*

*Proof.* Denote by  $a''$  the intersecting point of the segments  $ba'$  and  $\overline{ao}$ . Consider the convex hexagon  $\mathcal{H}_1 = ac'ba''cb'$ . Obviously, the hexagon  $\mathcal{H}_1 \subseteq \mathcal{H} = ac'ba'cb'$ . If  $|bb'|$  or  $|cc'|$  equals  $\min\{|aa''|, |bb'|, |cc'|\}$ , then, according to Lemma 2,  $\lambda_{\mathcal{H}}(ab')$  or  $\lambda_{\mathcal{H}}(ac')$  is at most  $8 - 4\sqrt{3}$ . If  $|aa''|$  equals  $\min\{|aa''|, |bb'|, |cc'|\}$ , then, by Lemma 2,  $|ca''|$  is at most  $8 - 4\sqrt{3}$ . Without loss of generality, we may assume that  $|aa'| = |bb'| = |aa''|$ , see Fig. 6. Then there exist points  $e$  and  $f$  such that  $be \parallel a''c$ ,  $bf \parallel a'c$ , and  $ef \parallel a'a''$ . Then,  $\triangle ca'a'' \sim \triangle bfe$ , hence,  $\frac{2|a''c|}{|be|} = \frac{2|a'c|}{|bf|} \leq 8 - 4\sqrt{3}$ . Since  $|bf|$  is smaller than the affine diameter parallel to  $a'c$ , therefore,  $\lambda_{\mathcal{H}}(a'c) \leq 8 - 4\sqrt{3}$ . The proof is complete.  $\square$

**Remark 2.** For the case  $a' \in \overline{ao}$ ,  $b' \notin \overline{bo}$ ,  $c' \in \overline{co}$ , or the case  $a' \in \overline{ao}$ ,  $b' \in \overline{bo}$ ,  $c' \notin \overline{co}$  the conclusion of Lemma 3 can be reached similarly.

**Lemma 4.** *If  $\triangle abc$  is a regular triangle,  $a' \notin \overline{ao}$ ,  $b' \notin \overline{bo}$ ,  $c' \in \overline{co}$  (see Fig. 7), then the convex hexagon  $\mathcal{H} = ac'ba'cb'$  has a side of relative length at most  $8 - 4\sqrt{3}$ .*

*Proof.* Denote by  $a''$  the intersecting point of  $ba'$  and  $\overline{ao}$ , and  $b''$  the intersecting point of  $cb'$  and  $\overline{bo}$ . If the hexagon  $\mathcal{H}_1 = ac'ba''cb''$  is a special-regular hexagon, then the points  $a$  and  $c'$  are distant in relative length by at most  $8 - 4\sqrt{3}$ . Otherwise, we have three cases to consider. When  $|cc'| = \min\{|aa''|, |bb''|, |cc'|\}$ , by Lemma 2 we obtain  $\lambda_{\mathcal{H}}(ac') \leq 8 - 4\sqrt{3}$ ; when  $|aa''| = \min\{|aa''|, |bb''|, |cc'|\}$ , then by Lemma 2 we have  $\lambda_{\mathcal{H}_1}(ca'') \leq 8 - 4\sqrt{3}$ , and so  $\lambda_{\mathcal{H}}(ca') \leq 8 - 4\sqrt{3}$ ; when  $|bb''| = \min\{|aa''|, |bb''|, |cc'|\}$ , then by Lemma 2 we have  $\lambda_{\mathcal{H}_1}(ab'') \leq 8 - 4\sqrt{3}$ , and so  $\lambda_{\mathcal{H}}(ab') \leq 8 - 4\sqrt{3}$ . The proof is complete.  $\square$

**Remark 3.** For the case  $a' \notin \overline{ao}$ ,  $b' \in \overline{bo}$ ,  $c' \notin \overline{co}$ , or the case  $a' \in \overline{ao}$ ,  $b' \notin \overline{bo}$ ,  $c' \notin \overline{co}$  the conclusion of Lemma 4 can be reached similarly.

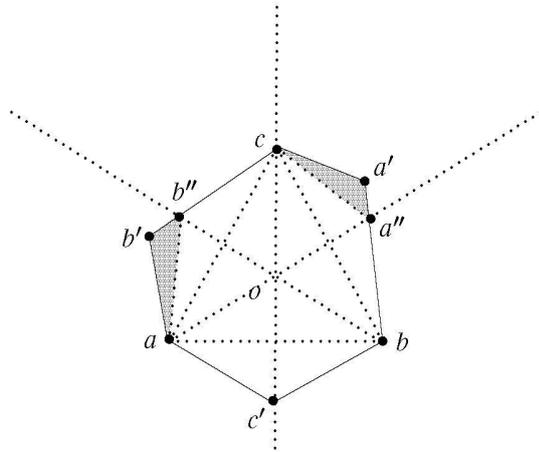


Fig. 7

**Lemma 5.** *If  $\triangle abc$  is a regular triangle,  $a' \notin \overline{ao}$ ,  $b' \notin \overline{bo}$ ,  $c' \notin \overline{co}$  (see Fig. 8), then the convex hexagon  $\mathcal{H} = ac'ba'cb'$  has a side of relative length at most  $8 - 4\sqrt{3}$ .*

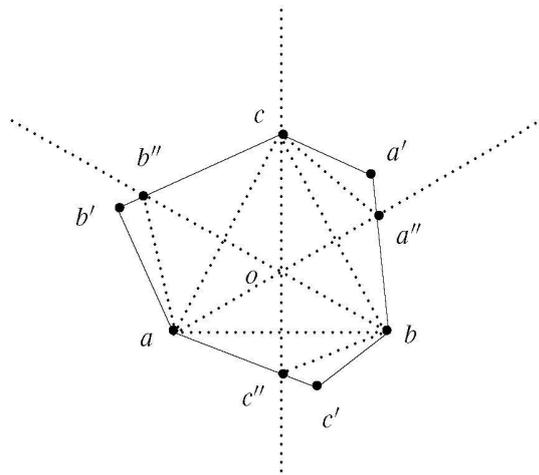


Fig. 8

*Proof.* Denote by  $a''$  the intersecting point of  $ba'$  and  $\overline{ao}$ ,  $b''$  the intersecting point of  $cb'$  and  $\overline{bo}$ , and  $c''$  the intersecting point of  $ac'$  and  $\overline{co}$ . Consider the hexagon  $\mathcal{H}_1 = ac''ba''cb''$ . Without loss of generality we may assume that  $|aa''| = \min\{|aa''|, |bb''|, |cc''|\}$ , then by Lemma 2 we have  $\lambda_{\mathcal{H}_1}(ca'') \leq 8 - 4\sqrt{3}$  and hence  $\lambda_{\mathcal{H}}(ca') \leq 8 - 4\sqrt{3}$ . The proof is complete.  $\square$

*Proof of Theorem 1.* Combining Lemmas 1–5 we obtain Theorem 1, that is, the conjecture of Doliwka and Lassak is true. By Remark 1 the upper bound is tight.  $\square$

From Remark 1 we know that many convex hexagons have relative long sides, but not every convex polygon has a relative long side. For example we have the following result:

**Theorem 2.** *There exists a convex 12-gon which does not have any relatively long side.*

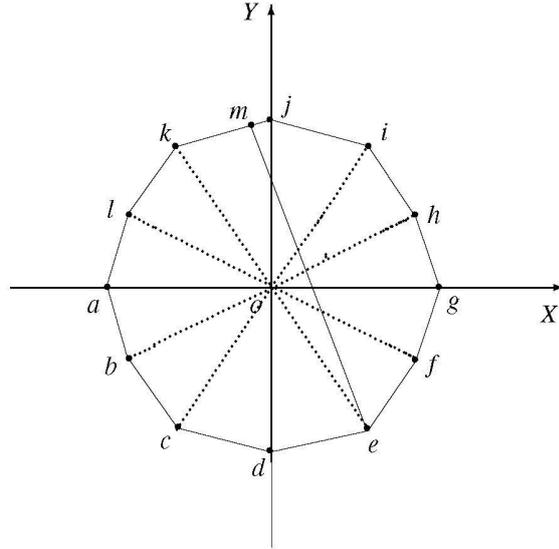


Fig. 9

*Proof.* We consider the convex 12-gon  $Q = abcdefghijkl$ , as shown in Fig. 9, where  $|oa| = |oc| = |oe| = |og| = |oi| = |ok| = t$  with  $1 \leq t < \frac{2\sqrt{3}}{3}$ ,  $|ob| = |od| = |of| = |oh| = |oj| = |ol| = 1$ , the angle formed by any two consecutive segments with common point  $o$  equals  $\frac{\pi}{6}$ . By the symmetry of the construction of the convex 12-gon all sides of the 12-gon have the same relative length. We need only to compute one of them, say,  $\lambda_Q(gh)$ . Obviously,  $e = (\frac{t}{2}, -\frac{\sqrt{3}t}{2})$ ,  $g = (t, 0)$ ,  $h = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ ,  $j = (0, 1)$ ,  $k = (-\frac{t}{2}, \frac{\sqrt{3}t}{2})$ , and there exists a point  $m \in jk$  such that the segments  $gh$  and  $em$  are parallel (see Fig. 9). The computation shows

$$|gh| = \sqrt{t^2 - \sqrt{3}t + 1}, \quad m = \left( \frac{\sqrt{3}t^3 - \sqrt{3}t}{2\sqrt{3}t^2 - 8t + 2\sqrt{3}}, \frac{-3t^3 + 4\sqrt{3}t^2 - 5t}{2\sqrt{3}t^2 - 8t + 2\sqrt{3}} \right),$$

hence,

$$|em| = \frac{2t\sqrt{t^2 - \sqrt{3}t + 1}}{|\sqrt{3}t^2 - 4t + \sqrt{3}|}.$$

Therefore, we obtain

$$\lambda_{\mathcal{Q}}(gh) = \frac{2|gh|}{|em|} = \frac{-\sqrt{3}t^2 + 4t - \sqrt{3}}{t} = 4 - \sqrt{3}\left(t + \frac{1}{t}\right) \leq 2(2 - \sqrt{3}).$$

However, we can easily obtain that  $\lambda_{12} = 2(2 - \sqrt{3})$  by setting  $t = 1$  in the above equation since  $\mathcal{Q}$  is a regular 12-gon when  $t = 1$ . Therefore,  $\lambda_{\mathcal{Q}}(gh) \leq \lambda_{12}$  and the proof is complete.  $\square$

### Acknowledgement

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