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Elemente der Mathematik

Affine regular polygons

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Sándor Szabó received his Ph.D. from Eötvös University in Budapest in 1980. He held visiting positions at the University of Dundee (UK), the University of California at Davis and the University of Pacific. Recently, he has been appointed as a faculty member to the University of Pécs, Hungary. His main mathematical interests lie in algebra and its applications to geometry and combinatorics.

1 Introduction

Let n, k be integers such that $n \geq 3$, $1 \leq k \leq n-1$, k is relatively prime to n. Let R be a counter-clockwise rotation about the point O by $k(360°/n)$. The images of the point P

$$Q_0 = R^0(P),\ Q_1 = R^1(P), \dots, Q_{n-1} = R^{n-1}(P)$$

are on one circle and divide the circle into n equal arcs. The directed straight line segments

$$\overrightarrow{Q_0Q_1},\ \overrightarrow{Q_1Q_2}, \dots, \overrightarrow{Q_{n-2}Q_{n-1}},\ \overrightarrow{Q_{n-1}Q_0}$$

are the sides of a regular (n, k)-gon (see Fig. 1). The regular $(n, 1)$-gon is an ordinary regular n-gon with directed sides. The regular $(n, n-1)$-gon is the same ordinary regular n-gon, only the orientation of the sides are the opposite.

For $2 \leq k \leq n-2$ a regular (n, k)-gon is a star polygon. An affine regular (n, k)-gon is an affine image of a regular (n, k)-gon (see Fig. 2). We will show that three results on triangles extend to affine regular polygons.

Drei nicht notwendigerweise gleich grosse gleichseitige Dreiecke mit einem gemeinsamen Eckpunkt heissen in Propelleranordnung. Verbindet man benachbarte, freie Ecken und bestimmt die Mittelpunkte der Verbindungsstrecken, so bilden diese Mitten ein gleichseitiges Dreieck. Diesen Propellersatz über Dreiecke verallgemeinert der Autor, indem er die gleichseitigen Dreiecke durch affine Bilder regulärer n-Ecke, die auch die Form eines Sternvielecks haben dürfen, ersetzt. Die Beweismethode beruht auf mehrfacher Anwendung von Rotationen, die mit der affinen Abbildung kommutieren, und auf geschickter Anwendung trigonometrischer Beziehungen. Ausserdem werden in dieser Arbeit der altbekannte Satz von Napoleon und der Satz von den Transversalen, die beide für Dreiecke gelten, mit ähnlichen Methoden auf affine Vielecke erweitert.

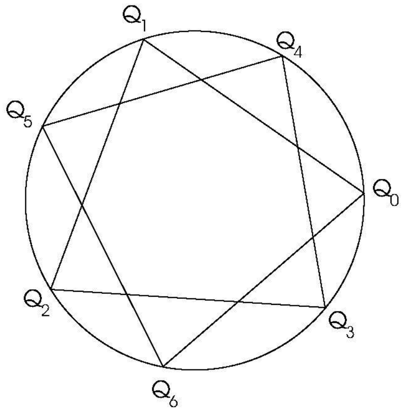

Fig. 1

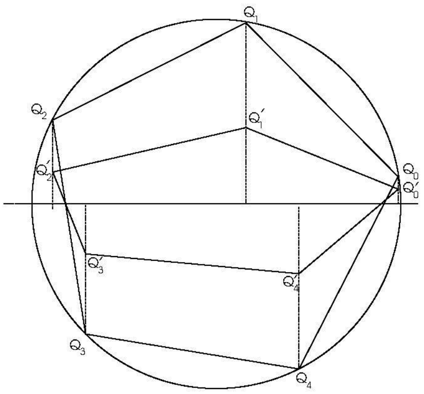

Fig. 2

2 The propeller theorem

L. Bankoff, P. Erdös, and M. Klamkin [1] proved the following result what they called the propeller theorem. Rotate a triangle about an arbitrarily chosen point by 60°. Let B_0, B_1, B_2 be the vertices of the original triangle, let C_0, C_1, C_2 be the vertices of the rotated triangle and let f be the cyclic permutation

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

of the indices 0, 1, 2. Then the midpoints D_0, D_1, D_2 of the sections

$$B_0C_{f(0)}, \quad B_1C_{f(1)}, \quad B_2C_{f(2)}$$

are the vertices of a regular triangle. The propeller theorem is a special case of the following theorem.

Theorem 1. *Rotate an affine regular (n, k)-gon about a point by $k(n-2)(180°/n)$. Let $B_0, B_1, \ldots, B_{n-1}$ be the vertices of the original affine regular (n, k)-gon, let $C_0, C_1, \ldots, C_{n-1}$ be the vertices of the rotated (n, k)-gon and let f be the cyclic permutation*

$$\begin{bmatrix} 0 & 1 & 2 & \ldots & n-2 & n-1 \\ n-1 & 0 & 1 & \ldots & n-3 & n-2 \end{bmatrix}$$

of the indices $0, 1, \ldots, n-1$. Then the midpoints $D_0, D_1, \ldots, D_{n-1}$ of the sections

$$B_0 C_{f(0)},\ B_1 C_{f(1)}, \ldots, B_{n-1} C_{f(n-1)}$$

are the vertices of a regular (n, k)-gon.

As a triangle is always an affine regular 3-gon Theorem 1 implies the propeller theorem. An affine regular 4-gon is a parallelogram so Theorem 1 is about rotating a parallelogram by 90°.

Proof. Let R be a rotation about the origin of the coordinate system by $k(180°/n)$ and let u be a vector. Clearly $R^0 u, R^2 u, \ldots, R^{2(n-1)} u$ are the vertices

$$A_0, A_1, \ldots, A_{n-1}$$

of a regular (n, k)-gon. If S is an affine transformation, then

$$SR^0 u + b, SR^2 u + b, \ldots, SR^{2(n-1)} u + b$$

are the vertices $B_0, B_1, \ldots, B_{n-1}$ of an affine regular (n, k)-gon. Furthermore,

$$\begin{gathered} R^{n-2} SR^0 u + R^{n-2} b, \\ R^{n-2} SR^2 u + R^{n-2} b, \\ \vdots \\ R^{n-2} SR^{2(n-1)} u + R^{n-2} b \end{gathered}$$

are the vertices $C_0, C_1, \ldots, C_{n-1}$ of the rotated copy of the (n, k)-gon $B_0 B_1 \ldots B_{n-1}$. The midpoints $D_0, D_1, \ldots, D_{n-1}$ of the sections

$$B_0 C_{f(0)},\ B_1 C_{f(1)}, \ldots, B_{n-1} C_{f(n-1)}$$

can be expressed in the following way:

$$\begin{gathered} \frac{1}{2}[SR^0 u + b + R^{n-2} SR^{2(n-1)} u + R^{n-2} b], \\ \frac{1}{2}[SR^2 u + b + R^{n-2} SR^0 u + R^{n-2} b], \\ \frac{1}{2}[SR^4 u + b + R^{n-2} SR^2 u + R^{n-2} b], \\ \vdots \end{gathered}$$

$$\frac{1}{2}[SR^{2(n-2)}u + b + R^{n-2}SR^{2(n-3)}u + R^{n-2}b],$$
$$\frac{1}{2}[SR^{2(n-1)}u + b + R^{n-2}SR^{2(n-2)}u + R^{n-2}b].$$

We want to show that

$$\begin{aligned} R^2(\overrightarrow{D_0D_1}) &= \overrightarrow{D_1D_2}, \\ R^2(\overrightarrow{D_1D_2}) &= \overrightarrow{D_2D_3}, \\ &\vdots \\ R^2(\overrightarrow{D_{n-1}D_0}) &= \overrightarrow{D_0D_1}. \end{aligned}$$

It is enough to verify that

$$\begin{aligned} R^2SR^0 + R^2R^{n-2}SR^{2(n-1)} &= SR^2 + R^{n-2}SR^0, \\ R^2SR^2 + R^2R^{n-2}SR^0 &= SR^4 + R^{n-2}SR^2, \\ R^2SR^4 + R^2R^{n-2}SR^2 &= SR^6 + R^{n-2}SR^4, \\ &\vdots \\ R^2SR^{2(n-2)} + R^2R^{n-2}SR^{2(n-3)} &= SR^{2(n-1)} + R^{n-2}SR^{2(n-2)}, \\ R^2SR^{2(n-1)} + R^2R^{n-2}SR^{2(n-2)} &= SR^0 + R^{n-2}SR^{2(n-1)}. \end{aligned}$$

Any of these equations is equivalent to

$$R^2S + SR^{n-2} = SR^2 + R^{n-2}S. \tag{1}$$

We prove this for the first and the last equations separately and for the remaining ones together. In the case of the first equation we get (1) in the following steps

$$\begin{aligned} R^2S + R^2R^{n-2}SR^{2(n-1)} &= SR^2 + R^{n-2}S, \\ R^2S + R^nSR^{n+n-2} &= SR^2 + R^{n-2}S, \end{aligned}$$

using that $R^n = -I$. For the last equation

$$\begin{aligned} R^2SR^{2(n-1)} + R^2R^{n-2}SR^{2(n-2)} &= SR^0 + R^{n-2}SR^{2(n-1)}, \\ -R^2SR^{n-2} + SR^{n-4} &= S - R^{n-2}SR^{n-2}. \end{aligned}$$

After multiplying by R^2 from the right we get (1). For the remaining equations

$$R^2SR^{2i} + R^2R^{n-2}SR^{2(i-1)} = SR^{2(i+1)} + R^{n-2}SR^{2i},$$

where $1 \le i \le n-2$,

$$R^2SR^{2i} - SR^{2i-2} = SR^{2i+2} + R^{n-2}SR^{2i}.$$

Multiplying by R^{n-2i} from the right we get (1).

So it remains to verify that (1) holds. For $0 \le t \le 1$ the affine transformation T with the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \tag{2}$$

shrinks the plane in the direction of the second coordinate axis. The image of a circle is an ellipse. We can get every possible shape of ellipses with a suitable choice of t. Consider now a regular n-gon together with the circle passing through the vertices. Applying first a rotation W to the regular n-gon then T we can get every possible shape of affine regular n-gons with a suitable choice of W and T. In short, we may represent the affine transformation S in the form TW. Using the fact that W and R commute the equation

$$R^2TW + TWR^{n-2} = TWR^2 + R^{n-2}TW$$

is equivalent to

$$R^2T + TR^{n-2} = TR^2 + R^{n-2}T. \tag{3}$$

The matrices of R^2, R^{n-2} are

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \quad \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix},$$

respectively, where $\alpha = 2k(180°/n)$, $\beta = (n-2)k(180°/n)$. Using these matrices and $\sin\alpha = \sin\beta$ it is a routine computation to verify that (3) holds. □

3 The Napoleon theorem

Consider a triangle with vertices B_0, B_1, B_2. Construct the triangles $B_0B_1B_0'$, $B_1B_2B_1'$, $B_2B_0B_2'$ such that they are regular and all are outside of the $B_0B_1B_2$ triangle. Then the centroids C_0, C_1, C_2 of the constructed triangles are vertices of a regular triangle. This result is known as Napoleon's theorem. Napoleon's theorem is a special case of the next theorem.

Theorem 2. *Consider an affine regular n-gon Π with vertices $B_0, B_1, \dots, B_{n-1}$. Erect regular n-gons on each side of Π such that all these are outside of Π. Then the centroids $C_0, C_1, \dots, C_{n-1}$ of these regular n-gons are the vertices of a regular n-gon.*

A triangle is always an affine regular triangle, so Theorem 2 is a generalization of Napoleon's theorem. An affine regular 4-gon is a parallelogram and a regular 4-gon is a square. So Theorem 2 is about erecting squares on the sides of a parallelogram.

Proof. If R is a rotation about the origin O by $90°/n$, S is an affine map and u is a vector, then

$$SR^0u, SR^4u, SR^8u, \dots, SR^{4(n-2)}u, SR^{4(n-1)}u$$

are the vertices $B_0, B_1, \dots, B_{n-1}$ of an affine regular n-gon Π.

Rotating B_0B_1 about B_0 by

$$360° - \left(90° - \frac{180°}{n}\right) = [4n - (n-2)]\left(\frac{90°}{n}\right) = (3n+2)\left(\frac{90°}{n}\right)$$

and multiplying it by

$$\lambda = \frac{1}{2\sin[2(90°/n)]}$$

we get the centroid of the regular n-gon erected on the side B_0B_1 (see Fig. 3).

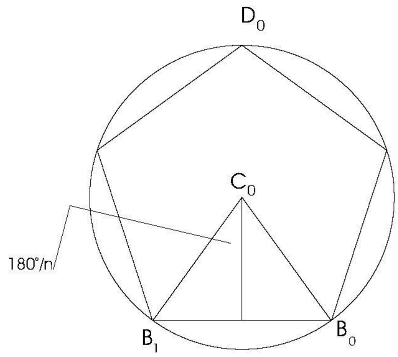

Fig. 3

Therefore the centroids $C_0, C_1, \ldots, C_{n-1}$ of the erected regular n-gons are

$$\begin{aligned}
&SR^0u + \lambda R^{3n+2}[SR^4 - SR^0]u,\\
&SR^4u + \lambda R^{3n+2}[SR^8 - SR^4]u,\\
&SR^8u + \lambda R^{3n+2}[SR^{12} - SR^8]u,\\
&\qquad\qquad\vdots\\
&SR^{4(n-2)}u + \lambda R^{3n+2}[SR^{4(n-1)} - SR^{4(n-2)}]u,\\
&SR^{4(n-1)}u + \lambda R^{3n+2}[SR^0 - SR^{4(n-1)}]u.
\end{aligned}$$

After setting $V = S + \lambda R^{3n+2}S[R^4 - I]$ we get that $C_0, C_1, \ldots, C_{n-1}$ are

$$VR^0u,\ VR^4u, \ldots, VR^{4(n-1)}u.$$

We want to show that

$$\begin{aligned}
R^4(\overrightarrow{OB_0}) &= \overrightarrow{OC_1},\\
R^4(\overrightarrow{OB_1}) &= \overrightarrow{OC_2},\\
&\ \vdots\\
R^4(\overrightarrow{OB_{n-2}}) &= \overrightarrow{OC_{n-1}},\\
R^4(\overrightarrow{OB_{n-1}}) &= \overrightarrow{OC_0}.
\end{aligned}$$

It is enough to check that

$$
\begin{aligned}
R^4 V R^0 &= V R^4, \\
R^4 V R^4 &= V R^8, \\
&\vdots \\
R^4 V R^{4(n-2)} &= V R^{4(n-1)}, \\
R^4 V R^{4(n-1)} &= V R^0.
\end{aligned}
$$

Any of these equations is equivalent to $R^4 V = V R^4$. As in the proof of Theorem 1 we may represent the affine transformation S in the form TW, where T is an affine transformation with a matrix (2) and W is a rotation. In the equation $R^4 V = V R^4$ the transformation V can be reduced to

$$V = T + \lambda R^{3n+2} T[R^4 - I]$$

as W commutes with rotations. Let the matrices of V, R^{3n+2}, R^4 be

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}, \quad \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix},$$

respectively. Here

$$
\begin{gathered}
\gamma = 4\Big(\frac{90^\circ}{n}\Big), \quad \beta = (3n+2)\Big(\frac{90^\circ}{n}\Big), \\
\gamma = 2\alpha, \quad \beta = 3 \cdot 90^\circ + \alpha.
\end{gathered}
$$

A routine computation shows that $R^4 V = V R^4$ is equivalent to $a = d$ and $b = -c$; furthermore

$$
\begin{aligned}
a &= \lambda \cos\beta(\cos\gamma - 1) - \lambda t \sin\beta \sin\gamma + 1, \\
b &= -\lambda \cos\beta \sin\gamma - \lambda t \sin\beta(\cos\gamma - 1), \\
c &= \lambda \sin\beta(\cos\gamma - 1) + \lambda t \cos\beta \sin\gamma, \\
d &= -\lambda \sin\beta \sin\gamma + \lambda t \cos\beta(\cos\gamma - 1) + t.
\end{aligned}
$$

The expressions for a, b, c, d are linear polynomials in t. Equating the like terms we have

$$\lambda \cos\beta(\cos\gamma - 1) + 1 = -\lambda \sin\beta \sin\gamma, \tag{4}$$

$$-\lambda t \sin\beta \sin\gamma = \lambda t \cos\beta(\cos\gamma - 1) + t, \tag{5}$$

$$\lambda \sin\beta(\cos\gamma - 1) = \lambda \cos\beta \sin\gamma, \tag{6}$$

$$\lambda t \cos\beta \sin\gamma = \lambda t \sin\beta(\cos\gamma - 1). \tag{7}$$

Clearly, it is enough to verify (4) and (6). Using

$$\sin\beta = -\cos\alpha, \quad \cos\beta = \sin\alpha, \quad \lambda = 1/(2\sin\alpha),$$

we can verify readily that (4) and (6) hold. □

4 A transversal theorem

Erect regular triangles $B_0B_1B_0'$, $B_1B_2B_1'$, $B_2B_0B_2'$ on the sides of a given $B_0B_1B_2$ triangle such that the new triangles are all outside of the old one. Then the straight line segments $B_0'B_2$, $B_1'B_0$, $B_2'B_1$ have the same lengths. The next theorem is a generalization of this result.

Theorem 3. *Consider an affine regular n-gon Π with vertices $B_0, B_1, \dots, B_{n-1}$. Erect regular n-gons Π_0, Π_1 on the sides B_0B_1, B_1B_2 of Π such that both Π_0 and Π_1 are outside of Π. If n is odd, then the distances of the vertices of Π_0, Π and Π_1, Π opposite to the sides B_0B_1 and B_1B_2 are equal. If n is even, then the distances of the midpoints of the edges of Π_0, Π and Π_1, Π opposite to the sides B_0B_1 and B_1B_2 are equal.*

Proof. Let R be a rotation about the origin O by $90°/n$, S an affine map, u a vector. Then $SR^0u, SR^4u, \dots, SR^{4(n-1)}u$ are the vertices $B_0, B_1, \dots, B_{n-1}$ of an affine regular n-gon Π.

Let us first deal with the $n = 2k + 1$ case. Rotating $\overrightarrow{B_0B_1}$ about B_0 by

$$360° - \left(90° - \frac{90°}{n}\right) = (3n + 1)\left(\frac{90°}{n}\right)$$

and multiplying it by

$$\lambda = \frac{1}{2\sin(90°/n)}$$

we get the vertex D_0 of Π_0 opposite to the B_0B_1 side (see Fig. 4).

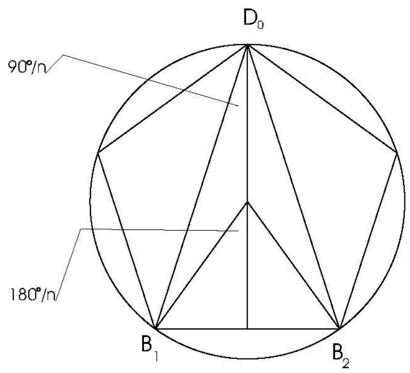

Fig. 4

Similarly rotating $\overrightarrow{B_1B_2}$ about B_1 by $(3n + 1)(90°/n)$ and multiplying it by λ we get the vertex D_1 of Π_1 opposite to the B_1B_2 side. As $n = 2k + 1$, the vertices B_{k+1}, B_{k+2} are the vertices of Π opposite to the sides B_0B_1, B_1B_2, respectively. The vertices D_0, D_1 are

$$SR^0u + \lambda R^{3n+1}[SR^4u - SR^0u],$$
$$SR^4u + \lambda R^{3n+1}[SR^8u - SR^4u].$$

We want to show that $R^4(\overrightarrow{B_{k+1}D_0}) = \overrightarrow{B_{k+2}D_1}$. Set

$$\begin{aligned} V &= S + \lambda R^{3n+1} S[R^4 - I] - SR^{4(k+1)} \\ &= S[I - R^{4(k+1)}] + \lambda R^{3n+1} S[R^4 - I]. \end{aligned}$$

It is enough to verify that $R^4 V = V R^4$. The affine map S can be written in the form $S = TW$, where W is a rotation and T is an affine map with a matrix (2). As rotations commute, V can be reduced to

$$V = T[I - R^{4(k+1)}] + \lambda R^{3n+1} T[R^4 - I].$$

Let the matrices of $R^{4(k+1)}$, R^{3n+1}, R^4, V be

$$\begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix}, \qquad \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix},$$

$$\begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

respectively. Here

$$\begin{aligned} \delta &= 4(k+1)\Big(\frac{90^\circ}{n}\Big) = 180^\circ + 2\Big(\frac{90^\circ}{n}\Big) = 180^\circ + 2\alpha, \\ \beta &= (3n+1)\Big(\frac{90^\circ}{n}\Big) = 3 \cdot 90^\circ + \Big(\frac{90^\circ}{n}\Big) = 270^\circ + \alpha, \\ \gamma &= 4\Big(\frac{90^\circ}{n}\Big) = 4\alpha. \end{aligned}$$

We can see that $R^4 V = V R^4$ is equivalent to $a = d$ and $b = -c$; furthermore

$$\begin{aligned} a &= \lambda\cos\beta(\cos\gamma - 1) - \lambda t \sin\beta \sin\gamma + 1 - \cos\delta, \\ b &= -\lambda\cos\beta\sin\gamma - \lambda t \sin\beta(\cos\gamma - 1) + \sin\delta, \\ c &= \lambda\sin\beta(\cos\gamma - 1) + \lambda t \cos\beta\sin\gamma - t\sin\delta, \\ d &= -\lambda\sin\beta\sin\gamma + \lambda t\cos\beta(\cos\gamma - 1) + t - t\cos\delta. \end{aligned}$$

The expressions for a, b, c, d are linear polynomials in t. Equating the like terms we have

$$\lambda\cos\beta(\cos\gamma + 1) - \cos\delta + 1 = -\lambda\sin\beta\sin\gamma, \tag{8}$$

$$-\lambda t\sin\beta\sin\gamma = \lambda t\cos\beta(\cos\gamma - 1) + t - t\cos\delta, \tag{9}$$

$$\lambda\sin\beta(\cos\gamma - 1) = \lambda\cos\beta\sin\gamma - \sin\delta, \tag{10}$$

$$\lambda t\cos\beta\sin\gamma - t\sin\delta = \lambda t\sin\beta(\cos\gamma - 1). \tag{11}$$

Clearly it is enough to verify (8) and (10). Note that

$$\begin{aligned} \sin\beta &= -\cos\alpha, \qquad \cos\beta = \sin\alpha, \\ \sin\delta &= -\sin 2\alpha, \qquad \cos\delta = -\cos 2\alpha, \\ \lambda &= 1/(2\sin\alpha). \end{aligned}$$

Substituting these into (8) and (10) we can verify that (8) and (10) hold.

Let us turn to the $n = 2k$ case. Rotating $\overrightarrow{B_0B_1}$ about B_0 by

$$360^\circ - 90^\circ = (3n)\Big(\frac{90^\circ}{n}\Big),$$

multiplying it by

$$\lambda = \frac{\cos\big(2(90^\circ/n)\big)}{\sin\big(2(90^\circ/n)\big)}$$

and adding $(1/2)\overrightarrow{B_0B_1} = u_0$ we get D_0, the midpoint of the side of Π_0 opposite to B_0B_1 (see Fig. 5).

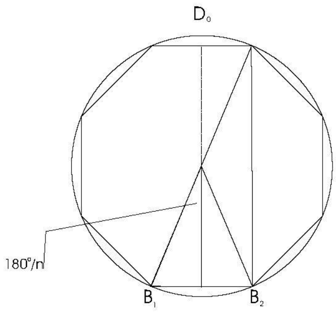

Fig. 5

Similarly, rotating $\overrightarrow{B_1B_2}$ about B_1 by $(3n)(90^\circ/n)$, multiplying it by λ and adding $(1/2)\overrightarrow{B_1B_2} = u_1$ we get D_1, the midpoint of the side of Π_1 opposite to B_1B_2. The vertices D_0, D_1 are

$$SR^0u + \lambda R^{3n}[SR^4u - SR^0u] + u_0,$$
$$SR^4u + \lambda R^{3n}[SR^8u - SR^4u] + u_1.$$

The midpoints E_0, E_1 of the sides of Π opposite to B_0B_1 and B_1B_2 are $B_{k+1} + u_0$ and $B_{k+2} + u_1$, respectively. We want to show that $R^4(\overrightarrow{E_0D_0}) = \overrightarrow{E_1D_1}$. Set

$$\begin{aligned} V &= S + \lambda R^{3n}S[R^4 - I] - SR^{4(k+1)} \\ &= S[I - R^{4(k+1)}] + \lambda R^{3n}S[R^4 - I]. \end{aligned}$$

We have to verify that $R^4V = VR^4$. We can represent S in the form $S = TW$, where W is a rotation and T has matrix (2). Now V reduces to $T[I - R^{4(k+1)}] + \lambda R^{3n}T[R^4 - I]$. The equations we have to verify are the same as earlier. Note that

$$\delta = 4(k+1)(90^\circ/n) = 180^\circ + 4(90^\circ/n) = 180^\circ + 2\alpha,$$

where

$$\alpha = 2(90°/n), \qquad \beta = (3n)(90°/n) = 270°,$$

$$\gamma = 4(90°/n) = 2\alpha, \qquad \lambda = \frac{\cos 2\big((90°/n)\big)}{\sin 2\big((90°/n)\big)} = \frac{\cos\alpha}{\sin\alpha}.$$

Using

$$\sin\delta = -\sin(2\alpha), \quad \cos\delta = -\cos(2\alpha)$$

we can verify the desired equations. □

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[1] Bankoff, L.; Erdös, P.; Klamkin, M.: The asymmetric propeller. *Math. Mag.* 46 (1973), 270–272.

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