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Triangle Centers Associated with Rhombi

Floor van Lamoen

Floor van Lamoen was born in 1966 in Leiden (NL). He studied mathematics at the University of Amsterdam and graduated in 1990. After his study he chose to become a high school teacher, his current profession. It was also after his study that he was attracted to the beauty of plane Euclidean and related geometry, and more specifically triangle geometry. Outside mathematics he is a national level racewalker. He is married and has one daughter.

Abstract

Not many of the famous triangle centers, or other central objects, are associated with rhombi, or even squares. In this paper four examples of such centers are given. They are related to well-known central objects in a triangle.

1 Introduction

Triangle geometry usually is about triangles, lines, circles, conic sections and cubic curves. Quadrilaterals do not to come into the picture very often. We can use rhombi however quite well to define pleasant triangle centers.

In the representation of triangle centers, we will make use of *homogeneous barycentric coordinates*, or shortly *barycentrics*, with respect to a fixed reference triangle ABC. The notion of these coordinates goes back to Möbius. Traditionally the point with coordinates (ξ, η, ζ) is considered as the center of mass (barycenter) of $\triangle ABC$ when located in

Wer glaubt, die Dreiecksgeometrie sei ein abgeschlossenes Gebiet, den lehrt dieser Beitrag, dass auch heute noch Neues und dabei auch Interessantes in der Elementargeometrie entdeckt werden kann – und das in der traditionellen Art und Weise, also ohne Einsatz eines Computers, d.h. ohne dynamische Geometriesoftware.

Im Falle ausgezeichneter Punkte eines Dreiecks haben wir den Umkreis- und Inkreismittelpunkt, den Höhenschnittpunkt (Orthozentrum) und den Schwerpunkt (Baryzentrum) noch aus der Schule in Erinnerung. In dem vorliegenden Beitrag nutzt F. van Lamoen die von A.F. Möbius eingeführten baryzentrischen Koordinaten bzw. trilinearen Koordinaten und stösst dabei auf interessante Rhomben und Quadrate, die sich so als ausgezeichnete Figuren des Dreiecks erweisen. *jk*

A, B and C are 'weights' of magnitudes ξ, η and ζ respectively (and the rest of the triangle is weightless).

More conveniently the equivalent definition is used, that (ξ, η, ζ) denotes the point P such that $Area(PBC) : Area(APC) : Area(ABP) = \xi : \eta : \zeta$. Here, we take $Area(PQR)$ to be zero when P, Q and R are collinear, positive when the orientation of the vertices of $\triangle PQR$ is counter-clockwise, and negative when the orientation is clockwise.

Barycentric coordinates are homogeneous in the sense that for any nonzero real number t the coordinates (ξ, η, ζ) and $(t\xi, t\eta, t\zeta)$ refer to the same point. For this reason we will write $(\xi : \eta : \zeta)$.

All coordinates in this paper will be barycentrics, unless mentioned otherwise.

Rather similar to barycentric coordinates is the notion of *trilinear coordinates* (trilinears). Trilinear coordinates $(\xi : \eta : \zeta)$ denote the point P with ratio of distances to the sidelines $d(P, \ell_{BC}) : d(P, \ell_{AC}) : d(P, \ell_{AB}) = \xi : \eta : \zeta$ (ℓ_{XY} denotes the line through X and Y). We give sign to, for instance, $d(P, \ell_{AB})$ in the same way as to $Area(ABP)$ in barycentric coordinates. Trilinear coordinates are homogeneous in the same sense as barycentrics. They are used and explained extensively in [2].

A simple relation between barycentric coordinates $(\xi_b : \eta_b : \zeta_b)$ and trilinear coordinates $(\xi_t : \eta_t : \zeta_t)$ for the same point P is

$$\xi_b : \eta_b : \zeta_b = a\xi_t : b\eta_t : c\zeta_t = \xi_t \sin\alpha : \eta_t \sin\beta : \zeta_t \sin\gamma, \tag{1}$$

where $a = |BC|$, $b = |AC|$, $c = |AB|$, $\alpha = \angle A$, $\beta = \angle B$ and $\gamma = \angle C$.

In both coordinate systems a line ℓ is given by the equation $lx + my + nz = 0$. We usually write this as $\ell = [l : m : n]$, and observe that these *line coordinates* are homogeneous. Treating lines, as well as points, as vectors we find the line $\ell_{P_1P_2}$ as external product $\ell = P_1 \times P_2$. Dually, the point of intersection P of lines ℓ_1 and ℓ_2 is found by $P = \ell_1 \times \ell_2$.

A notion from triangle geometry that we will use several times in this paper is *isogonal conjugacy*. Two points P and Q are called *isogonal conjugates* when $\angle BAP = \angle QAC$, $\angle CBP = \angle QBA$ and $\angle ACP = \angle QCB$. Algebraicly the isogonal conjugate of $P = (\xi : \eta : \zeta)$, not on the sidelines of $\triangle ABC$, is $Q = (a^2/\xi : b^2/\eta : c^2/\zeta)$.

In this paper four ways will be presented to define triangle centers associated with rhombi. Points $\mathcal{A}_\psi$, $\mathcal{B}_\psi$, $\mathcal{C}_\psi$ and $\mathcal{D}_\psi$ will be *(triangle) centers* following the definition by Clark Kimberling in [2].

Kimberling's notion of *center* depends on the definition of *center functions*. Let $f(a, b, c)$ be a nonzero continuous function defined on triples (a, b, c), representing the sidelengths of the reference triangle. Function f is a center function

- if f is homogeneous, i.e. $\exists r \forall k : f(ka, kb, kc) = k^r f(a, b, c)$;
- and if f is symmetric in the sense that $f(a, b, c) = f(a, c, b)$.

Any point of the form $(\, f(a,b,c) : f(b,c,a) : f(c,a,b) \,)$ (either in trilinears, or in barycentrics) is called a *center*. Hereby a center is not a point in the classical sense, but it is seen as a function of the triangle it is defined in.

The first rhombi-center $\mathscr{A}_\psi$ will be associated with rhombi circumscribing $\triangle ABC$, $\mathscr{C}_\psi$ with rhombi inscribed in $\triangle ABC$, $\mathscr{B}_\psi$ and $\mathscr{D}_\psi$ will be defined by three congruent rhombi. The centers will be depending on the internal angles ψ and $\bar{\psi} := \pi - \psi$ of the rhombi. Here $\psi(A, B, C)$ is considered to be independent from permutation of (A, B, C).

Applying more of Clark Kimberling's definitions, centers $\mathscr{A}_\psi$ and $\mathscr{B}_\psi$ for fixed ψ will be *major centers* (see [1]), since these centers can be written as $(f(\alpha) : f(\beta) : f(\gamma))$ for some continuous function f. Finally, $\triangle A'B'C'$ of Section 2 will be a *central triangle* (see [3] for the least complicated definition).

2 Rhombi circumscribing a triangle

The definition of the first triangle center $\mathscr{A}_\psi$ depends on the following:

Theorem 1 *Let $\triangle ABC$ and $\psi \in (-\pi, \pi) \setminus \{0\}$ be given. We can construct a unique rhombus $AXA'Y$ such that $\angle XAY = \psi$, $B \in \ell_{XA'}$ and $C \in \ell_{A'Y}$. Similarly we can construct rhombi to find points B' and C'. The constructed $\triangle A'B'C'$ is perspective to $\triangle ABC$, i.e. $\ell_{AA'}$, $\ell_{BB'}$ and $\ell_{CC'}$ are concurrent.*

The perspector (center of perspectivity) $\ell_{AA'} \cap \ell_{BB'} \cap \ell_{CC'}$ will be called the *circumrhombi point* $\mathscr{A}_\psi$.

Proof. Denote by $\rho_{\chi,X}(P)$ the image of P after rotation through χ about X. Let $B'' = \rho_{\bar{\psi},A}(B)$ and let $C'' = \rho_{-\bar{\psi},A}(C)$. Then let $\{A'\} = \ell_{B''C} \cap \ell_{C''B}$. Points $X \in \ell_{C''A'}$ and $Y \in \ell_{B''A'}$ can be constructed in such a way that $AXA'Y$ is a parallelogram (see Figure 1).

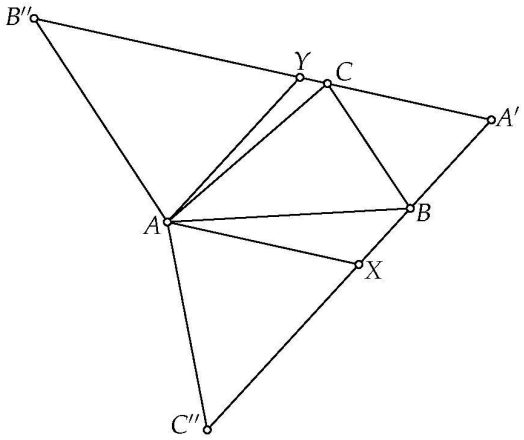

Fig. 1 Construction of rhombus $AXA'Y$ circumscribing $\triangle ABC$ for $\psi = \frac{\pi}{3}$.

Observe that $\triangle AC''B \cong \triangle ACB''$, so the perpendicular distances from A to $\ell_{B''A'}$ and $\ell_{C''A'}$ are equal. And $AXA'Y$ must be a rhombus.

Note that $\rho_{\bar{\psi},A}(\ell_{C''XBA'}) = \ell_{A'CYB''}$, so $\angle YA'X = \psi$ (regarding orientation). And $AXA'Y$ is a rhombus satisfying the requirements of the theorem.

It is easy to see that $AXA'Y$ is the unique rhombus fulfilling these requirements: When we rotate the complete figure of $\triangle ABC$ and rhombus $AXA'Y$ through $\bar{\psi}$ about A, and let B'' be the image of B again, we immediately see that $B'' \in \ell_{A'C}$. In the same way we see that the image of C after rotation through $-\bar{\psi}$ about A must be on the line $\ell_{A'B}$.

To prove the last statement of the theorem, we use barycentrics. First we calculate barycentrics for C''. To do this, let C_a'', C_b'' and C_c'' be the orthogonal projections of C'' on sidelines ℓ_{BC}, ℓ_{AC} and ℓ_{AB} respectively. We will calculate the trilinear distances $\xi := \pm|C''C_a''|$, $\eta := \pm|C''C_b''|$ and $\zeta := \pm|C''C_c''|$, given signs in the usual way for trilinear coordinates.

Straightforwardly we find that $\eta = b\sin\psi$ and $\zeta = -b\sin(\bar{\psi} - \alpha) = -b\sin(\alpha + \psi)$. Using that $\triangle C''AC$ is isosceles, we see $|C''C| = 2b\cos(\frac{\psi}{2})$, and consequently $\xi = 2b\cos(\frac{\psi}{2})\sin(\gamma - \frac{\psi}{2})$. With use of (1) this brings as barycentrics for C'':

$$C'' = \left(2a\cos\left(\frac{\psi}{2}\right)\sin\left(\gamma - \frac{\psi}{2}\right) : b\sin\psi : -c\sin(\alpha + \psi)\right).$$

In the same way B'' becomes:

$$B'' = \left(2a\cos\left(\frac{\psi}{2}\right)\sin\left(\beta - \frac{\psi}{2}\right) : -b\sin(\alpha + \psi) : c\sin\psi\right).$$

Straightforward calculations give that barycentrics for $\{A'\} = \ell_{BB''} \cap \ell_{CC''}$ are:

$$A' = \left(-2a\csc(\alpha + \psi) : b\sec\left(\frac{\psi}{2}\right)\csc\left(\beta - \frac{\psi}{2}\right) : c\sec\left(\frac{\psi}{2}\right)\csc\left(\gamma - \frac{\psi}{2}\right)\right).$$

With similar results for B' and C', we conclude that $\ell_{AA'}$, $\ell_{BB'}$ and $\ell_{CC'}$ concur in

$$\mathcal{A}_\psi = \left(\sin\alpha\csc\left(\alpha - \frac{\psi}{2}\right) : \sin\beta\csc\left(\beta - \frac{\psi}{2}\right) : \sin\gamma\csc\left(\gamma - \frac{\psi}{2}\right)\right). \qquad \square$$

All circumrhombi points lie on the Kiepert hyperbola, a rectangular hyperbola consisting of the isogonal conjugates of points on the Brocard axis (the line through the circumcenter and the symmedian or Lemoine point). See also Section 3.

For $\psi = \pm\frac{2\pi}{3}$ we find that $\triangle A'B'C'$ degenerates into a single point. The two points are the isogonic centers (X_{13} (the Fermat-Torricelli point) and X_{14} in [2]).

Among the huge list of triangle centers in [2] there are quite a few centers that can be found as circumrhombi points. Examples are the Napoleon points, the third Brocard point and the Tarry point. Points that might be added are the *circumsquares points*, found when $\psi = \pm\frac{\pi}{2}$.

3 Congruent rhombi points

Let $\psi \in [-\pi, \pi]$. We call $\mathcal{B}_\psi$ a *congruent rhombi point* if there exist six points $P_A, Q_C \in \ell_{AC}$, $P_C, Q_B \in \ell_{BC}$ and $Q_A, P_B \in \ell_{AB}$ such that:

- These six points are on a circle with center $\mathcal{B}_\psi$;
- $\angle P_A\mathcal{B}_\psi Q_A = \angle P_B\mathcal{B}_\psi Q_B = \angle P_C\mathcal{B}_\psi Q_C = \psi$.

Triangles $P_A\mathcal{B}_\psi Q_A$, $P_B\mathcal{B}_\psi Q_B$ and $P_C\mathcal{B}_\psi Q_C$ are congruent and form the halves of three congruent rhombi: $P_A\mathcal{B}_\psi Q_A A'$ (the *A-rhombus*), $P_B\mathcal{B}_\psi Q_B B'$ and $P_C\mathcal{B}_\psi Q_C C'$ (see Figure 2).

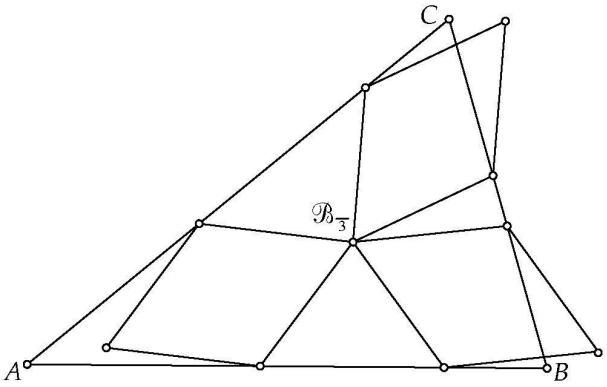

Fig. 2 The congruent rhombi point $\mathcal{B}_{\frac{\pi}{3}}$.

This way to define a triangle center is inspired by the *Kenmotu point* or *congruent squares point*, found in Kimberling's list of triangle centers in [2] as X_{371}, which point is $\mathcal{B}_{\frac{\pi}{2}}$. The Kenmotu point, named after the author of a Japanese mathematics problem collection in 1840, is the only center in this list, which is given a description directly using squares or rhombi. The point $\mathcal{B}_{-\frac{\pi}{2}}$ is mentioned in [2] as X_{372}, but without definition using squares directly.

Theorem 2 *The congruent rhombi point $\mathcal{B}_\psi$ is the point with barycentrics*

$$\left(\sin\alpha\cos\left(\alpha - \frac{\psi}{2}\right) : \sin\beta\cos\left(\beta - \frac{\psi}{2}\right) : \sin\gamma\cos\left(\gamma - \frac{\psi}{2}\right)\right).$$

Proof. Suppose the point $\mathcal{B}_\psi$ and the points P_A, P_B, P_C, Q_A, Q_B and Q_C from the definition exist. Let $\chi_a := \angle Q_B\mathcal{B}_\psi P_C$, $\chi_b := \angle Q_C\mathcal{B}_\psi P_A$ and $\chi_c := \angle Q_A\mathcal{B}_\psi P_B$. The circle with $A\mathcal{B}_\psi$ as diameter passes through the midpoints of Q_AP_B and P_AQ_C. From this and similar results for the circles with diameters $B\mathcal{B}_\psi$ and $C\mathcal{B}_\psi$ we find the following system of equations:

$$\frac{\chi_a}{2} + \frac{\chi_b}{2} + \psi = \alpha + \beta, \quad \frac{\chi_b}{2} + \frac{\chi_c}{2} + \psi = \beta + \gamma, \quad \frac{\chi_a}{2} + \frac{\chi_c}{2} + \psi = \alpha + \gamma.$$

This gives $\chi_a = 2\alpha - \psi$, $\chi_b = 2\beta - \psi$ and $\chi_c = 2\gamma - \psi$. Since χ_a, χ_b and χ_c are the top angles of three isosceles triangles on the sides of $\triangle ABC$ with congruent legs, $\mathcal{B}_\psi$ must have barycentrics as in the theorem.

On the other hand, we can construct a point $\mathcal{B}_\psi$ with barycentrics as in the theorem when ψ is given: $\mathcal{B}_\psi$ is the isogonal conjugate of $\mathcal{A}_{-\bar{\psi}}$ when $\psi \in (-\pi, \pi) \setminus \{0\}$, the circumcenter when $\psi = 0$ and the symmedian point when $\psi = \pm\pi$. Having $\mathcal{B}_\psi$, we can also construct the points P_A, P_B, P_C, Q_A, Q_B and Q_C using the results from the first part of the proof. □

All congruent rhombi points $\mathcal{B}_\psi$ lie on the Brocard axis. Well-known centers that can be found as $\mathcal{B}_\psi$ are the isodynamic points and the Brocard midpoint.

It is not difficult to see that $\triangle ABC$ and $\triangle A'B'C'$ (A' was the vertex of the A-inrhombus opposite to $\mathcal{B}_\psi$; B' and C' were found similarly) are homothetic (parallel). Let ξ be the perpendicular distance between BC and $B'C'$, given sign in the usual way for trilinear coordinates w.r.t. $\triangle ABC$, and let η and ζ be defined likewise. Then straightforward computation gives us that

$$\xi : \eta : \zeta = \cos\left(\alpha + \frac{\psi}{2}\right) : \cos\left(\beta + \frac{\psi}{2}\right) : \cos\left(\gamma + \frac{\psi}{2}\right).$$

We can apply the following lemma:

Lemma 3 *Let $\triangle ABC$ and $\triangle A'B'C'$ be homothetic. Let ξ, η and ζ be the distances between the a-, b- and c-sides of these triangles respectively, given signs in the usual way for trilinear coordinates w.r.t. $\triangle ABC$. Then the perspector of $\triangle ABC$ and $\triangle A'B'C'$ has trilinears $(\xi : \eta : \zeta)$ w.r.t. both triangles.*

The proof of this lemma is easy and left to the reader.

Using Lemma 3 we find that the perspector of $\triangle ABC$ and $\triangle A'B'C'$ is $\mathcal{B}_{-\psi}$.

Consequently, but this is also seen directly from angle computations using earlier results on the top angles of triangles such as $\triangle Q_C \mathcal{B}_\psi P_A$, each pair of A-, B- and C-rhombi for ψ and $-\psi$ must be parallel. See Figure 3.

4 Inrhombi points

The easiest way to find a triangle center using rhombi seems to be the use of rhombi inscribed in $\triangle ABC$.

Given $\triangle ABC$, we can compute the sidelength of a rhombus $TUVW$ with $\angle TUV = \psi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$ in such a way that these conditions hold: $\ell_{TU} = \ell_{AB}$, $V \in \ell_{BC}$, $W \in \ell_{AC}$ and points V and W lie on the same side of ℓ_{AB} as C. We will call rhombus $TUVW$ the *C-inrhombus*. After similarly having found the A- and B-inrhombi, ℓ_{VW} is the c-side of a $\triangle A'B'C'$ which is homothetic to $\triangle ABC$. The perspector $\mathcal{C}_\psi$ of $\triangle A'B'C'$ and $\triangle ABC$ will be called the *inrhombi point*.

To calculate the sidelengths of $TUVW$ let $p := |VW|/c = |VC|/a = |WC|/b$. We find that:

$$|VW| = pc, \tag{2}$$

$$|VU| = \frac{(1-p)a \sin\beta}{\sin\psi}. \tag{3}$$

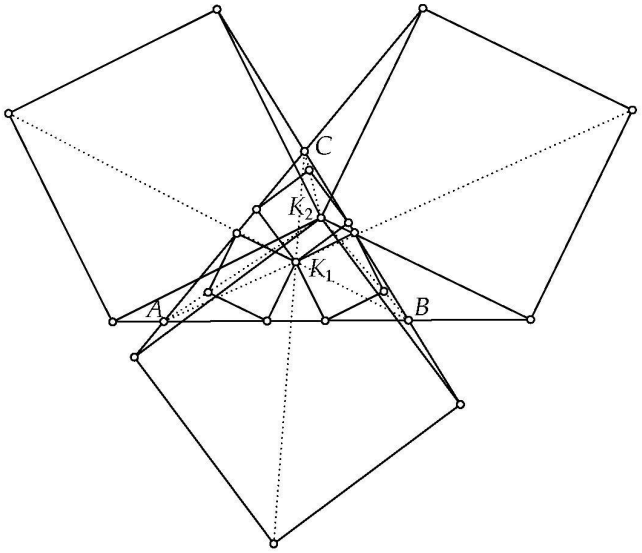

Fig. 3 Parallel squares and the two congruent squares points K_1 and K_2.

Then (2) = (3) gives

$$p = \frac{a \sin \beta}{a \sin \beta + c \sin \psi}. \tag{4}$$

Substituting (4) into (2) we see that

$$|VW| = \frac{2\sigma}{a \sin \beta + c \sin \psi}, \tag{5}$$

where σ denotes the area of $\triangle ABC$. Achieving similar results for A- and B-inrhombi, and using Lemma 3, we conclude that barycentrics for $\mathscr{C}_\psi$ are

$$\left(\frac{\sin \alpha}{\sin \beta \sin \gamma + \sin \alpha \sin \psi} : \frac{\sin \beta}{\sin \alpha \sin \gamma + \sin \beta \sin \psi} : \frac{\sin \gamma}{\sin \alpha \sin \beta + \sin \gamma \sin \psi} \right).$$

All inrhombi points lie on the hyperbola formed by the isogonal conjugates of points on the line through the centroid and the symmedian point of $\triangle ABC$.

5 Congruent shrinked inrhombi points

For $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, let $B_A, A_B \in \ell_{AB}$, $A_C, C_A \in \ell_{AC}$, $B_C, C_B \in \ell_{BC}$ and $\mathscr{D}_\psi$ fulfil the following statements:

- $\ell_{B_A C_A} \parallel \ell_{BC}$, $\ell_{A_B C_B} \parallel \ell_{AC}$ and $\ell_{A_C B_C} \parallel \ell_{AB}$;
- $\ell_{B_A C_A} \cap \ell_{A_B C_B} \cap \ell_{A_C B_C} = \{\mathscr{D}_\psi\}$;
- The A-inrhombus of $\triangle AB_A C_A$, the B-inrhombus of $\triangle A_B B C_B$ and the C-inrhombus of $\triangle A_C B_C C$ are congruent.

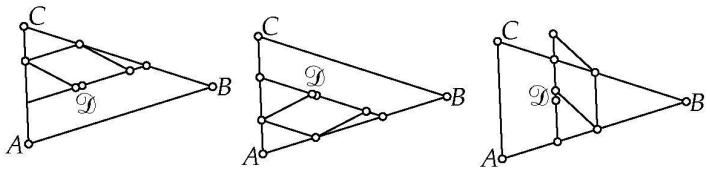

Fig. 4 The congruent shrinked inrhombi point $\mathfrak{D}$ for $\psi = \frac{\pi}{4}$.

$\mathfrak{D}_\psi$ will be called the *congruent shrinked inrhombi point*. See Figure 4.

To calculate coordinates for (possible) $\mathfrak{D}_\psi$ let $q_a := |B_A C_A|/|BC|$ and let q_b and q_c be defined likewise. To make the shrinked inrhombi congruent, we derive from (5) and similar results on sidelengths of the A- and B-inrhombi that there must be a real number t, such that $q_a = t(b\sin\gamma + a\sin\psi)$, $q_b = t(a\sin\gamma + b\sin\psi)$ and $q_c = t(a\sin\beta + b\sin\psi)$.

It is also easy to see that $Area(\triangle BC\mathfrak{D}_\psi) = (1-q_a)\sigma$, $Area(\triangle AC\mathfrak{D}_\psi) = (1-q_b)\sigma$ and $Area(\triangle AB\mathfrak{D}_\psi) = (1-q_c)\sigma$. The sum of these three areas is σ, so $q_a + q_b + q_c = 2$ and consequently

$$t = \frac{2}{b\sin\gamma + a\sin\gamma + a\sin\beta + (a+b+c)\sin\psi}.$$

And we find that $\mathfrak{D}_\psi$ has barycentrics

$$\begin{aligned}\big(&\sin\alpha(\sin\beta+\sin\gamma) - \sin\beta\sin\gamma + \sin\psi(\sin\beta+\sin\gamma-\sin\alpha) :\\ &\sin\beta(\sin\alpha+\sin\gamma) - \sin\alpha\sin\gamma + \sin\psi(\sin\alpha+\sin\gamma-\sin\beta) :\\ &\sin\gamma(\sin\alpha+\sin\beta) - \sin\alpha\sin\beta + \sin\psi(\sin\alpha+\sin\beta-\sin\gamma)\big).\end{aligned}$$

The congruent shrinked inrhombi points all lie on the line through the *Nagel point* and the *equal parallelians point* (X_{192} in [2], found when ψ=0). Only four centers on this line are mentioned in [2]. The *congruent shrinked insquares points* might be interesting additions to these.

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