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Planar Rectangular Sets and Steiner Symmetrization

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1 Introduction

Let K be a closed convex set in the plane. In [1], Danzer establishes the following pretty result.

Theorem 1. *If no rectangle inscribed in K has exactly three of its vertices on the boundary of K , then K is a circular disk.*

We generalize Danzer's characterization in the following way. Let OX , OY be given, fixed orthogonal axes in the plane. We say that K is a *rectangular set* if no inscribed rectangle with edges parallel to the given axes has exactly three of its vertices on the boundary of K . Some anomalies can occur in this new setting. For example, if K has two adjacent perpendicular edges which are parallel to the axes, there is an infinite number of 'inscribed' rectangles having just three vertices on the boundary of K . We therefore interpret *inscribed* here to imply that the given rectangle is the largest in the family of homothetic rectangles having vertices on the boundary of K . This is the assumption we would make if talking about an incircle of K .

We now ask if it is possible to characterize in some way the family \mathcal{R} of rectangular sets. We note that \mathcal{R} contains sets which are symmetric about either or both of the axes.

Let K be a closed convex set in the plane, and OX , OY given, fixed orthogonal axes. We say that K is a *rectangular set* if no inscribed rectangle with edges parallel to the given axes has exactly three of its vertices on the boundary of K . We show that if S_X , S_Y denote Steiner symmetrizations about the axes OX , OY respectively, then K is a rectangular set (relative to these axes) if and only if $S_X S_Y(K) = S_Y S_X(K)$. *psc*

It turns out that the family \mathcal{R} has a nice characterization in terms of Steiner symmetrization, which we now define. Let OA be a given line – the *axis* l of symmetrization. For each point p on OA let $u(p)$ be the line through p which is perpendicular to l . The set $u(p) \cap K$ is either the empty set, a point, or a line segment. If it is the empty set, we define $B(p)$ to be the empty set. If it is a point, we define $B(p)$ to be the point p . If it is a line segment, we define $B(p)$ to be the segment of equal length whose midpoint is p and which lies on $u(p)$. We now define K_A by

$$K_A = \cup_{p \in l} B(p).$$

The process of obtaining K_A from K in this way is called *Steiner symmetrization* about the line OA . Properties of this well-known and useful form of symmetrization can be found, for example, in Eggleston [2].

We shall establish the following connection between Steiner symmetrization and the family \mathcal{R} of rectangular sets.

Theorem 2. Let S_X, S_Y denote symmetrizations about the axes OX, OY respectively. Then K is a rectangular set (relative to these axes) if and only if

$$S_X S_Y(K) = S_Y S_X(K).$$

2 Proof of Theorem 2

For consistency in naming in the proof, we drop the function notation used in the statement of the theorem, and use $S_X S_Y$, for example, to mean first apply S_X and then apply S_Y . We shall also use the words *horizontal* and *vertical* to describe lines which are parallel to OX, OY respectively.

First we suppose that K is a rectangular set. Let A be a point on the boundary of K . By assumption, A will be a vertex of a (perhaps degenerate) rectangle $ABCD$ whose four vertices lie on the boundary of K (see Figure 1).

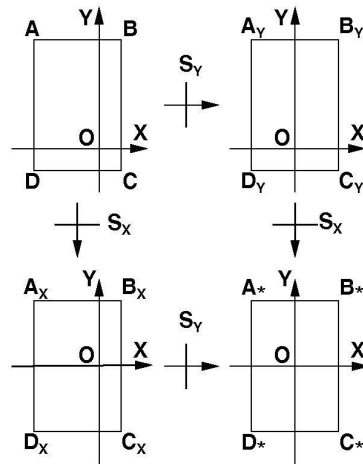


Fig. 1

Let $AB = 2x$ and $BC = 2y$. If we symmetrize K using S_Y to obtain a symmetrized set K_Y , then A will map to a point A_Y , a vertex of a rectangle $A_Y B_Y C_Y D_Y$, inscribed in K_Y , and congruent to $ABCD$. For, under the symmetrization, lengths AB, DC are preserved, and the image segments $A_Y B_Y, D_Y C_Y$ are centred on the axis OY . In particular, A_Y has x -coordinate x , and $A_Y D_Y = 2y$. If we now symmetrize K_Y using S_X to obtain set K_{YX} , then A_Y maps to a point A_{YX} , a vertex of a rectangle inscribed in K_{YX} and congruent to $ABCD$. For, under the symmetrization, lengths $A_Y D_Y, B_Y C_Y$ are preserved, and the image segments $A_{YX} D_{YX}, B_{YX} C_{YX}$ are centred on the axis OX . In particular, A_{YX} has x -coordinate x , and y -coordinate y .

It is clear from the symmetry of X and Y in this argument that the image of A under the product $S_X S_Y$ will be $A_{XY} = A_{YX} (= A_*$ in Figure 1). We deduce that $K_{XY} = K_{YX}$.

Now let us suppose that K is a set which has the same image under $S_Y S_X$ as it does under $S_X S_Y$. Thus $K_{YX} = K_{XY}$. We wish to show that K is a rectangular set. We observe that it will be sufficient to establish this result for the case when K is a polygon. The general case will then follow using a standard approximation argument. We may thus assume that the final symmetrized set $K_{XY} = K_{YX}$ is the convex hull of a finite family of rectangles having horizontal and vertical edges. If each of these rectangles occurs as the image of an inscribed rectangle in K , then K is a rectangular set, and there is nothing to prove. Suppose then that one of these rectangles, $R_{XY} = R_{YX}$ does not occur in this way. Let this rectangle have horizontal and vertical dimensions $2x, 2y$ respectively. Suppose too that y is the largest number for which this happens.

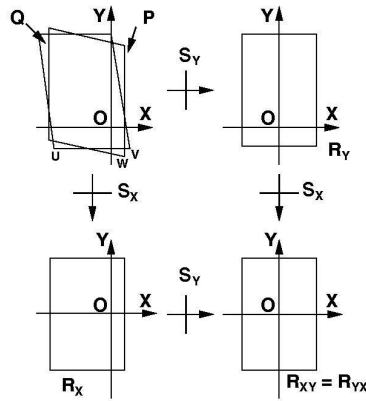


Fig. 2

Now R_{XY} is the image under S_Y of a set R_X (see Figure 2). In fact R_X is itself a rectangle, since it is inscribed in a set K_X which is symmetric about the X -axis. Further, R_X has horizontal and vertical dimensions $2x, 2y$ respectively. Now rectangle R_X occurs as the image under symmetrization S_X of a set P inscribed in the original set K . By the properties of symmetrization, this set P must be a parallelogram having one pair of vertical parallel edges. The length of each of these parallel edges is $2y$, and the

distance between them is $2x$. In the same way, R_{XY} occurs as the image under $S_Y S_X$ of a parallelogram Q inscribed in K having two horizontal parallel edges; the length of each of these parallel edges is $2x$, and the distance between them is $2y$.

If either of P, Q is a rectangle, then P, Q will coincide, as we have already seen that the image of a rectangle inscribed in K having horizontal and vertical edges is the same under the two successive symmetrizations, no matter which order of symmetrization is used. Hence parallelogram P extends strictly above or below the parallel horizontal edges of parallelogram Q . Inverting the figure if necessary, we may assume that P extends strictly below Q . Let UV denote the bottom horizontal edge of Q , labelled as in Figure 2, and W the vertex of P which lies below it. Then points U, W, V lie in an anti-clockwise order on the boundary of K . Since symmetrization is a continuous transformation, U, W, V will map under the successive symmetrizations S_Y, S_X to image points U^*, W^*, V^* lying in anti-clockwise order on the boundary of K_{XY} . But U^*V^* is the bottom edge of R_{XY} . It follows that W^* is the vertex of a rectangle inscribed in K_{XY} which does not arise as the image of a rectangle inscribed in K . Further, the vertical dimension of this rectangle exceeds the vertical dimension $2y$ of R_{XY} which was chosen to be maximal. This contradiction establishes the theorem.

3 Final Comment

The class of rectangular sets appears naturally here in terms of successive orthogonal symmetrizations; to my knowledge, this class does not occur elsewhere in the literature. It would be interesting to investigate whether this class of sets has other special properties.

References

- [1] Danzer, L. W., "A characterization of the circle", *Proc. Symp. Pure. Maths* VII (1963), 99–100.
- [2] Eggleston, H. G., *Convexity*, Cambridge Tract No. 47 (1963), Cambridge University Press.

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