El. Math. *51* (1996)

0013-6018/96/040156-05$1.50 + 0.20/0

An Analogue of the Pythagorean Theorem with Regular n-Gons instead of Squares

Darko Veljan

Darko Veljan was born in Zagreb. He began his studies at the University of Zagreb where he received his MSc; later he went to the United States and obtained his PhD from Cornell University (Ithaca, NY) in 1979. He is now Professor at the University of Zagreb. His interests include algebraic topology, combinatorics, and, especially, elementary geometry and mathematics education. He is (co-)author of several textbooks. In his spare time he likes mountaineering; he hopes to be able to climb to the top of some more of the Bosnian mountains after the war has ended and the mines have been cleared away.

Recall the famous Pythagorean theorem. Let $\triangle ABC$ be a right-angled triangle with side lengths a, b (legs) and hypotenuse c. Let F_a, F_b, and F_c be the areas of the squares erected over the sides. Then we have (Fig. 1):

$$F_c = F_a + F_b\,, \tag{1}$$

or

$$c^2 = a^2 + b^2\,. \tag{2}$$

This theorem is known for about 2 500 years and up to now it has more than 300 different proofs. One of the simplest and most popular proofs is shown in Fig. 2. Here F is the area of the triangle $\triangle ABC$. In fact, it is clear that the Pythagorean theorem is equivalent to the fact

$$F_{a+b} = F_c + 4F\,. \tag{3}$$

In words we can phrase (3) as follows. The area of a square over the sum of the legs of a right triangle is equal to the sum of four areas of the triangle and the area of a square over the hypotenuse.

Der Satz von Pythagoras ist mehr als 2500 Jahre alt. Ist es ein Zeichen ewiger Jugend, wenn er auch heute noch zu weiterführenden Überlegungen anregt und Nachkommen zeugt? Darko Veljan präsentiert in seinem Beitrag zwei neue Pythagoras-Abkömmlinge.

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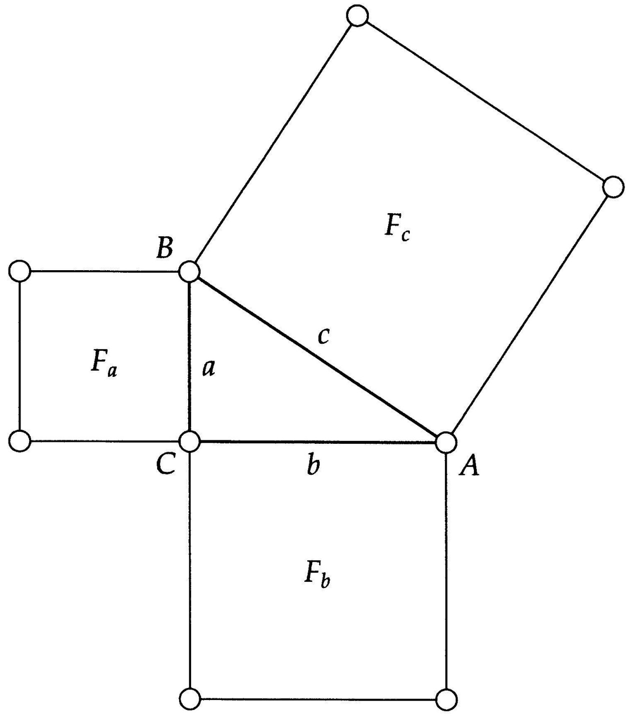

Abb. 1

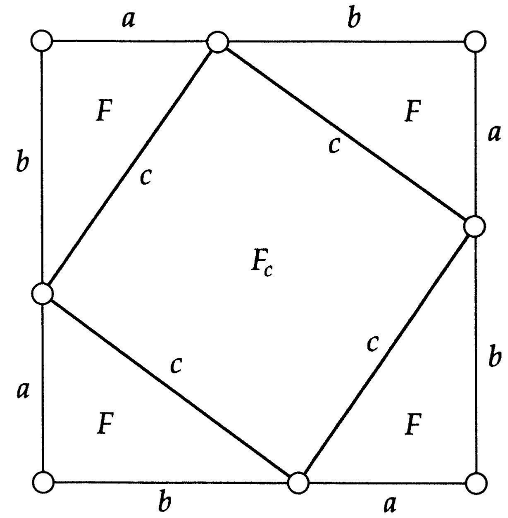

Abb. 2

In this paper we shall consider a triangle with an angle equal to the fraction $2\pi/n$, $n \geq 3$, or $\pi - \frac{2\pi}{n}$ and erect regular n-gons over its sides (or the sum of two sides) and prove relationships analogous to (1), (2) or (3).

So, let us fix an integer $n \geq 3$, and let $\triangle ABC$ be a triangle with side lengths a, b and c, angle at C equal to $\gamma = 2\pi/n$, and with area F. Further, for any $x > 0$, denote by $F_x^{(n)}$ the area of a regular n-gon with side-length equal to x.

Now consider the regular n-gon with side length equal to $a+b$, subdivide the sides such that the segments of lengths a and b alternate, and then join the consecutive subdivision points as in Fig. 2 for $n = 4$, in Fig. 3 for $n = 6$, or in Fig. 4 for $n = 3$.

Denote by c' the side length of the inscribed regular n-gon obtained in this way. Considering the areas, it is clear that the big n-gon consists of the inscribed regular n-gon with side length c' and of n copies of triangles of area F. Hence, we have

$$F_{a+b}^{(n)} = F_{c'}^{(n)} + nF\,. \tag{4}$$

The side c' can be obtained from our original triangle by reflecting the vertex A to A' over C; then $A'B = c'$ (see Fig. 5).

It is easily seen (e.g. by applying the law of cosine to $\triangle ABC$ and $\triangle A'BC$, or without trigonometry) that

$$c'^2 = 2(a^2 + b^2) - c^2\,. \tag{5}$$

Now consider a triangle with an angle $\gamma = \pi - 2\pi/n$. Then by the same construction as before we obtain instead of (4), the even simpler relation (see Fig. 6 for $n = 5$)

$$F_{a+b}^{(n)} = F_c^{(n)} + nF\,. \tag{6}$$

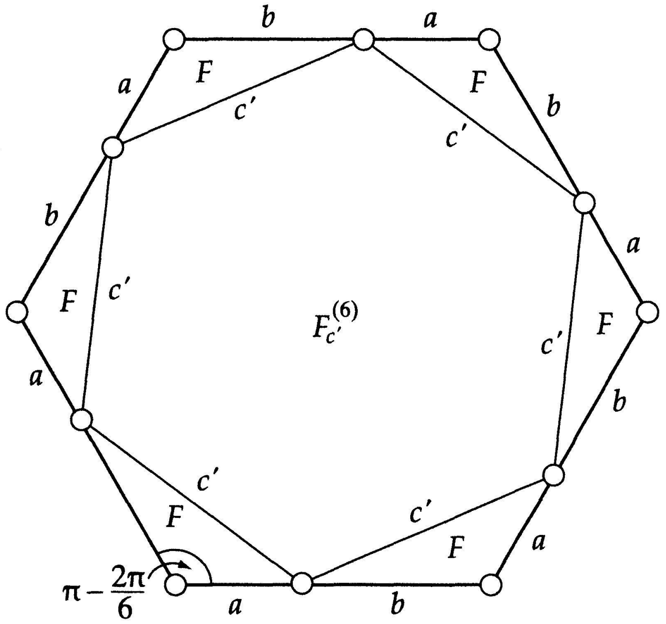

Abb. 3 $(n = 6)$

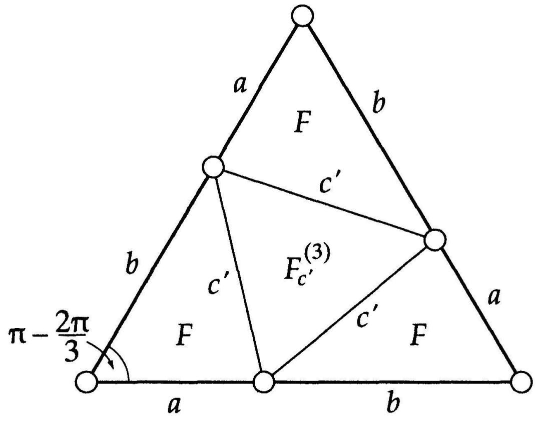

Abb. 4 $(n = 3)$

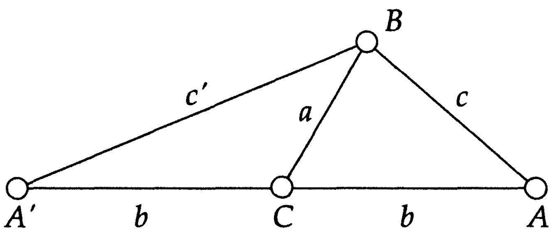

Abb. 5

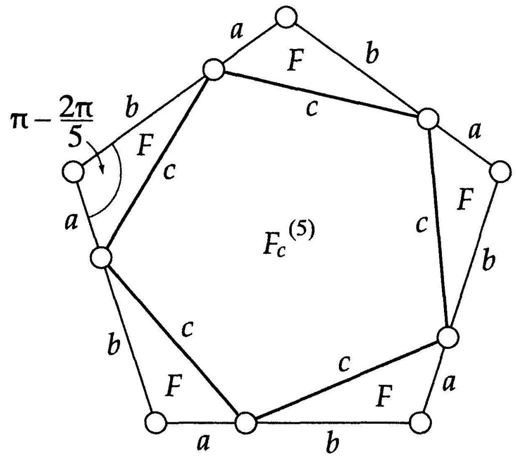

Abb. 6 $(n = 5)$

Let us adopt the following concepts. Call a triangle $\triangle ABC$ with angle $\gamma = 2\pi/n$ at C a *fraction n-triangle* and one with $\gamma = \pi - 2\pi/n$ a *complementary fraction n-triangle*; call the opposite side c its *hypotenuse*, and c' (given by (5)) its *complementary hypotenuse*, while the other two sides we call *legs*.

The relationships (4) and (6) can now be phrased in the following way.

Pythagorean theorem with regular n-gons . *Let $\triangle ABC$ be a fraction n-triangle (resp. complementary fraction n-triangle). Then the area of a regular n-gon over the sum of the legs is equal to the sum of n areas of the triangle and the area of a regular n-gon over the complementary hypotenuse (resp. hypotenuse).*

Note that a fraction n-triangle is congruent to its complementary triangle if and only if $n = 4$, and in this case both (4) and (6) reduce to (3). Therefore, this theorem can be considered as an n-gon analogue of the Pythagorean theorem.

Now let us find the right analogues of relations (1) and (2). As Pythagoras measured areas in square units, accordingly we shall measure areas by regular n-gon units. So, assume that the regular n-gon with side equal to 1 has area equal to 1. Then by the similarity argument, it follows that the area of a regular n-gon with side-length x is equal to x^2 (Fig. 7).

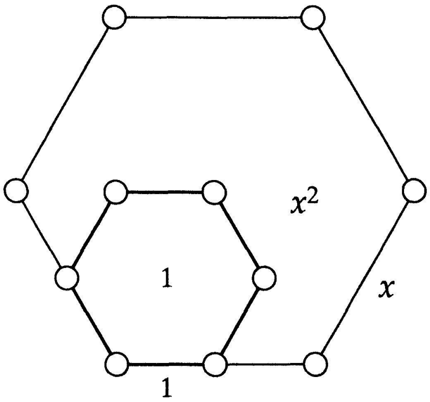

Fig. 7

So, let $G_x^{(n)}$ be the area of a regular n-gon with side length x, measured in n-gon units, and let $G^{(n)}$ be the area of a fraction n-triangle measured in n-gon units. Then (4) can be written in the form

$$G_{a+b}^{(n)} = G_{c'}^{(n)} + nG^{(n)},$$

and since $G_x^{(n)} = x^2$, from (5) we get

$$c^2 = a^2 + b^2 + nG^{(n)} - 2ab, \tag{7}$$

or

$$G_c^{(n)} = G_a^{(n)} + G_b^{(n)} + nG^{(n)} - 2ab. \tag{8}$$

One could also write (7) in the form

$$G_c^{(n)} = G_{|a-b|}^{(n)} + nG^{(n)}. \tag{9}$$

Similarly, for a complementary fraction n-triangle, (6) can be written as

$$G_{a+b}^{(n)} = G_c^{(n)} + nG^{(n)}, \tag{10}$$

and then in turn we get

$$c^2 = a^2 + b^2 + 2ab - nG^{(n)}, \tag{7'}$$

$$G_c^{(n)} = G_a^{(n)} + G_b^{(n)} + 2ab - nG^{(n)}. \tag{8'}$$

The relations (7) and (7′) are analogues of (2), while (8) and (8′) are analogues of (1). These relations can also be phrased in a similar fashion. Just as an illustration of these formulas, erect to the outside of the triangle regular n-gons over its sides, as in Fig. 8 for $n = 3$, and in Fig. 9 for $n = 6$.

The ordinary area (in square units) $F_x^{(n)}$ of a regular n-gon with side length x is given by

$$F_x^{(n)} = \frac{nx^2}{4} \cot \frac{\pi}{n}.$$

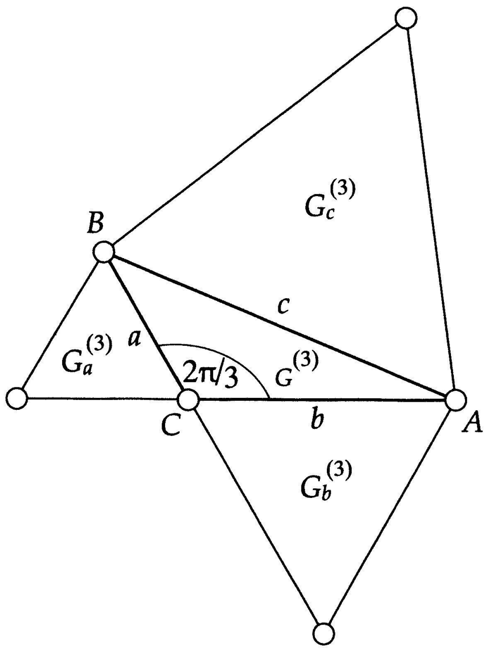

Abb. 8 $(n = 6)$

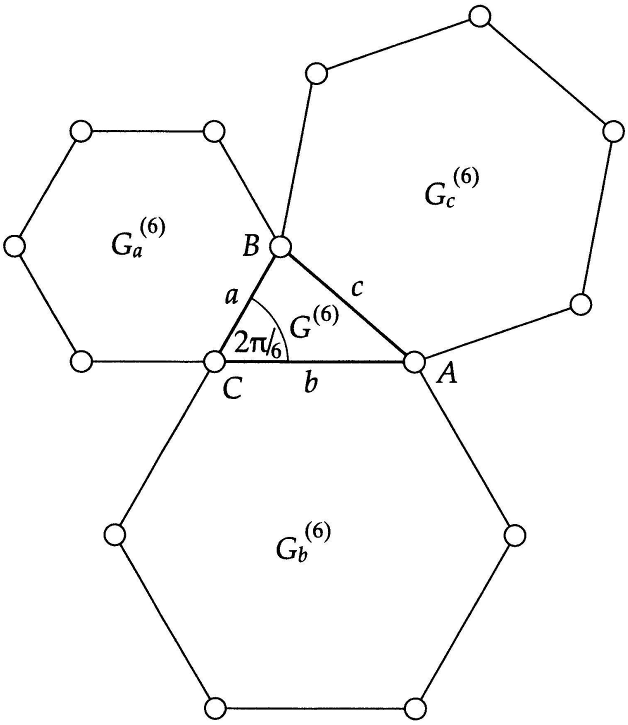

Abb. 9 $(n = 3)$

Since this area measured by n-gon unit is $G_x^{(n)} = x^2$, it follows that the scaling factor for areas is given by $F_x^{(n)} \big/ G_x^{(n)} = \frac{n}{4} \cot \frac{\pi}{n}$. Hence, for our triangle we have

$$\frac{1}{2} ab \sin \frac{2\pi}{n} = F = G^{(n)} \frac{n}{4} \cot \frac{\pi}{n}, \quad \text{which implies} \quad G^{(n)} = \frac{4ab}{n} \sin^2 \frac{\pi}{n}.$$

By substituting this into (7) or (7′) we would, of course, get the law of cosine for our $\triangle ABC$. However, the point here was to give a purely geometric and visual interpretation of the presented formulas in terms of areas.

In our previous paper [1] we have treated only the case $n = 3$. The approach there was slightly different from the one presented here.

It would be of interest to consider the above construction for any rational angle $\frac{m}{n} 2\pi$, and for affine regular polygons. It would also be interesting to find spherical and hyperbolical counterparts to our results.

References

[1] J. Kovačević, D. Veljan, A triangular analogue of the Pythagorean theorem, The Mathematical Gazette, to appear.

Darko Veljan
Department of Mathematics
University of Zagreb
10000 Zagreb, Bijenička c. 30
Croatia