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An elementary proof of the convergence of iterated exponentials

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Helmut Länger, born in 1951, obtained his doctoral degree in mathematics at the University of Technology in Vienna as a student of Professor Nöbauer. After spending a year in Darmstadt he returned to Vienna in 1977 where he is now working as an associate professor at the Institute of Algebra and Discrete Mathematics of the University of Technology. In the beginning of his career his main research interest was algebra, but in recent years he has also done work on discrete problems arising in biology and in axiomatic quantum mechanics.

What are the solutions of the equations

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = \frac{1}{2} \text{ or } x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2,$$

respectively (cf. [2])? Questions of such a type lead in a natural way to the problem of determining the convergence behaviour of the sequence

$$a, a^a, a^{a^a}, \dots \tag{1}$$

for a positive real number a. Here we are concerned with an iteration process with starting point $a > 0$ and iteration function $f(x) := a^x$.

Für welche Werte von a ist der iterative Prozess gegeben durch $x \longrightarrow a^x$ stabil? Für welche Werte von a ist die Folge

$$a, a^a, a^{a^a}, a^{a^{a^a}}, \dots$$

konvergent? Und was ist der Limes? Einfache, reizvolle Fragen mit einfachen, reizvollen Antworten! ***ust***

Local stability analysis of the iteration process $x \mapsto a^x$ $(a > 0)$. Since f is continuous, the possible (finite) limits of (1) must be fixed points of f. Put $g(x) := x^{\frac{1}{x}}$ for all $x > 0$. Then $\lim_{x\to 0} g(x) = 0$, g is strictly increasing on $(0, e]$, g is strictly decreasing on $[e, \infty)$ and $\lim_{x\to\infty} g(x) = 1$. Put $h_1 := (g\,|\,(0, e])^{-1}$ and $h_2 := (g\,|\,[e, \infty))^{-1}$. It is easy to see that the fixed points of f are exactly the positive real numbers b with $g(b) = a$. Moreover, $f'(b) = \ln b$ for every fixed point b of f. Hence we have the following six cases:

(i) If $a \in (0, \frac{1}{e^e})$ then $h_1(a) \in (0, \frac{1}{e})$ is the unique fixed point of f and $h_1(a)$ is unstable.

(ii) If $a = \frac{1}{e^e}$ then $h_1(a) = \frac{1}{e}$ is the unique fixed point of f.

(iii) If $a \in (\frac{1}{e^e}, 1]$ then $h_1(a) \in (\frac{1}{e}, 1]$ is the unique fixed point of f and $h_1(a)$ is asymptotically stable.

(iv) If $a \in (1, e^{\frac{1}{e}})$ then there exist exactly two fixed points of f, namely $h_1(a) \in (1, e)$ and $h_2(a) \in (e, \infty)$, $h_1(a)$ is asymptotically stable and $h_2(a)$ is unstable.

(v) If $a = e^{\frac{1}{e}}$ then $h_1(a) = h_2(a) = e$ is the unique fixed point of f.

(vi) If $a \in (e^{\frac{1}{e}}, \infty)$ then f has no fixed point.

Now we prove the following

Theorem. *Let a be a positive real number. Then the sequence (1) converges iff $a \in [\frac{1}{e^e}, e^{\frac{1}{e}}]$. In this case the corresponding limit is $h_1(a)$.*

Proof. In the following let x, y, z denote (arbitrary) positive real numbers. Put $a_1 := a$, $a_2 := a^a$, $a_3 := a^{a^a}$, ... Define $f^2(x) := f(f(x))$. Then we have

$$(f^2)'(x) = a^{a^x} a^x \ln^2 a \quad \text{and} \quad (f^2)''(x) = a^{a^x} a^x (\ln^3 a)(a^x \ln a + 1).$$

Observe that in case $x < y$ we have

$$\begin{aligned} &x^z < y^z \\ &z^x > z^y \quad \text{if} \quad z < 1 \\ &z^x < z^y \quad \text{if} \quad z > 1\,. \end{aligned}$$

If $a \in (1, e^{\frac{1}{e}}]$ then f is strictly increasing and hence

$$1 < a_1 < a_2 < a_3 < \dots\,.$$

Moreover, $a_n \le e$ for all n (which can be proved by induction on n), and therefore (1) converges to $h_1(a)$.

Now consider the only non-trivial remaining case $a \in (0, 1)$. Since $a^a < 1^a = 1$, $h_1(a) < 1$ and f is strictly decreasing, we have

$$a_1 < a_3 < a_5 < \dots < h_1(a) < \dots < a_6 < a_4 < a_2. \tag{2}$$

If $a \in (0, \frac{1}{e^e})$ then $f'(h_1(a)) < -1$ and hence (1) is divergent. If $a \in [\frac{1}{e^e}, 1)$ then we have

$$-1 = \ln \frac{1}{e} = \ln h_1(\frac{1}{e^e}) \leq \ln h_1(a) = a^{h_1(a)} \ln a < a^u \ln a < 0$$

for all $u > h_1(a)$ and hence $(f^2)'' < 0$ on $(h_1(a), \infty)$. Therefore

$$0 < (f^2)'(u) < (f^2)'(h_1(a)) = \ln^2 h_1(a) \leq 1$$

for all $u > h_1(a)$ which shows that the sequence a_2, a_4, a_6, ... and therefore also (1) converges to $h_1(a)$.

Remarks.

(i) The sequence $\frac{1}{4}, \frac{1}{4}^{\frac{1}{4}}, \frac{1}{4}^{\frac{1}{4}^{\frac{1}{4}}}, \ldots$ converges to $\frac{1}{2}$, i. e. $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = \frac{1}{2}$ has the unique solution $x = \frac{1}{4}$.

(ii) The sequence $\sqrt{2}, \sqrt{2}^{\sqrt{2}}, \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}, \ldots$ converges to 2 (cf. [2]), i. e. $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$ has the unique solution $x = \sqrt{2}$ which is somewhat surprising.

(iii) If (1) contains an element $\geq e$ then (1) is divergent.

(iv) If $a \in (0, \frac{1}{e^e})$ then (1) converges to a limit cycle of order 2.

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