Zeitschrift: Elemente der Mathematik

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 50 (1995)

Artikel: On the dynamic behaviour of Chebyshev polynomials

Autor: Länger, Helmut

DOI: https://doi.org/10.5169/seals-46341

El. Math. *50* (1995) 0013-6018/95/010028-03$1.50 + 0.20/0

On the Dynamic Behaviour of Chebyshev Polynomials

Helmut Länger

Helmut Länger, born in 1951, obtained his doctoral degree in mathematics at the University of Technology in Vienna as a student of Professor Nöbauer. After spending a year in Darmstadt he returned to Vienna in 1977 where he is now working as a lecturer at the Institute of Algebra and Discrete Mathematics of the University of Technology. In the beginning of his career his main research interest was algebra, but in recent years he has also done work on discrete problems arising in biology and in axiomatic quantum mechanics.

During the last years chaos theory has become more and more important. So-called "deterministic chaos" sometimes occurs in connection with certain iteration processes of the form $x_{k+1} = f(x_k)$, $k \geq 0$, where f is a mapping from an interval I of the real line into itself and $x_0 \in I$. So, for instance, the iteration process induced by the Feigenbaum function $f(x) = 4x(1-x)$ on $[0, 1]$ shows such a "chaotic" behaviour. This means that the sequence defined above often behaves like a sequence of random numbers. This property of f was investigated in [1]. Since f is a conjugate of the Chebyshev polynomial T_2 of second degree ($f = g \circ T_2 \circ g^{-1}$ with $g(x) = (1 - x)/2$), the results on f can be transformed into results on T_2. The aim of this paper is to generalize the main results of [1] concerning the "irregular" behaviour of the iteration process induced by T_2 to Chebyshev polynomials T_n of degree $n > 2$. For every positive integer k we determine the number of k-periodic points of T_n and show that these points are "unstable".

Wieder einmal treffen in diesem Beitrag zwei auf den ersten Blick völlig getrennte Gebiete und Begriffswelten der Mathematik aufeinander: Einerseits die Feigenbaumfunktion $f(x) = 4x(1-x)$ und ihr chaotisches Verhalten, andererseits die Chebyshev-Polynome $T_n(x)$, $n \geq 0$, die durch die Gleichung $T_n(\cos x) = \cos(nx)$ festgelegt sind und die u.a. in der Theorie der Orthogonalreihen eine grosse Rolle spielen. Die Verbindung wird hergestellt durch die Bemerkung, dass die Polynome f und T_2 zueinander konjugiert sind: es gibt eine Funktion g mit $f = g \circ T_2 \circ g^{-1}$. Fragen wie diese liegen dann auf der Hand: Zeigen die höheren Chebyshev-Polynome ebenfalls chaotisches Verhalten, und wenn ja, was ist zum Beispiel die Anzahl der k-periodischen Punkte von T_n? In seinem Beitrag gelingt es Helmut Länger, diese und eine Anzahl weiterer Fragen zu beantworten. *ust*

Let n be a non-negative integer. Then there exists exactly one function $T_n : [-1,1] \to [-1,1]$ satisfying $T_n(\cos x) = \cos nx$ for all real x. T_n is a polynomial of degree n (this follows from $T_0(x) = 1$, $T_1(x) = x$ and $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for all $n > 1$), the so-called Chebyshev polynomial of degree n.

In the following let n, k be positive integers and assume $n > 1$.

Definition Let M be a set and $f : M \to M$. A point a is called *k-periodic* for f if $f^k(a) = a$ and $f^i(a) \neq a$ for all $i = 1, \ldots, k-1$ (here and in the following f^i denotes the i-th iteration of f).

Theorem 1 *T_n has exactly*

$$\sum_{I \subseteq \{1,\ldots,r\}} (-1)^{|I|}\, n^{k/\prod_{\iota\in I} p_\iota}$$

k-periodic points where $p_1, \ldots, p_r$ are the different prime factors of k. (As usual, the "empty" product is defined to be 1.)

Proof. T_n has exactly n^k points with a period dividing k, namely 1, $\cos(2m\pi/(n^k+1)), m = 1, \ldots, n^k/2$, and $\cos(2m\pi/(n^k-1)), m = 1, \ldots, n^k/2 - 1$, if n is even, and 1, -1, $\cos(2m\pi/(n^k+1)), m = 1, \ldots, (n^k-1)/2$, and $\cos(2m\pi/(n^k-1)), m = 1, \ldots, (n^k-3)/2$, if n is odd. The rest follows from the inclusion-exclusion-principle.

Remark

(i) The continous mapping $T_n : [-1,1] \to [-1,1]$ has a 3-periodic point, namely $\cos(2\pi/(n^3+1))$. Hence by a celebrated result due to Li and Yorke (cf. [4]; for an easy proof of this result cf. [2]) it follows that T_n has at least one k-periodic point for every positive integer k. In fact, $\cos(2\pi/(n^k+1))$ is a k-periodic point of T_n for every positive integer k.

(ii) From the proof of Theorem 1 it follows that every element of $[-1,1]$ is the limit of a sequence of points of a period dividing $1, 2, 3, \ldots$, respectively.

Theorem 2

(i) *1 is a 1-periodic point of T_n and $T_n'(1) = n^2$.*

(ii) *-1 is a periodic point of T_n iff n is odd. In this case -1 is a 1-periodic point of T_n and $T_n'(-1) = n^2$.*

(iii) *If $a \in (-1,1)$ is a k-periodic point of T_n then $|({T_n}^k)'(a)| = n^k$.*

Proof. (i) and (ii) follow by differentiating $T_n(\cos x) = \cos nx$ twice.

Now let $a \in (-1,1)$ be a k-periodic point of T_n. From (i) and (ii) it follows that $|{T_n}^i(a)| \neq 1$ for all positive integers i. By differentiating $T_n(\cos x) = \cos nx$ $(x \in \mathbb{R})$ one obtains $-\sin x\, T_n'(\cos x) = -n \sin nx$ for all $x \in \mathbb{R}$ whence $(1-\cos^2 x)(T_n'(\cos x))^2 = n^2(1-(T_n(\cos x))^2)$ for all real x. This shows $(({T_n}^k)'(a))^2 = ({T_n}'({T_n}^{k-1}(a)) \cdot \ldots \cdot {T_n}'(a))^2 = ({T_n}'({T_n}^{k-1}(a)))^2 \cdot \ldots \cdot ({T_n}'(a))^2 = n^2(1-(T_n({T_n}^{k-1}(a)))^2) \cdot (1-({T_n}^{k-1}(a))^2)^{-1} \cdot \ldots \cdot n^2(1-(T_n(a))^2)(1-a^2)^{-1} = n^{2k}$.

Remark

(i) Theorem 2 shows that the periodic cycles of T_n are unstable.

(ii) The proof of the corresponding theorem in [1] cannot be generalized to Chebyshev polynomials of higher degree. Therefore it was necessary to find a completely new proof. (That $(1-x^2)(T_n{}'(x))^2 = n^2(1-(T_n(x))^2)$ holds for all $x \in [-1, 1]$ was already proved in [3].)

(iii) $(T_n^k)'\ (a) = n^k$ or $-n^k$ if a is of the form $\cos\big(2m\pi/(n^k-1)\big)$ or $\cos\big(2m\pi/(n^k+1)\big)$, respectively.

References

[1] P. Dormayer: Chaos bei der Feigenbaumabbildung $f(x) = 4x(1-x)$. Didaktik Math. **19** (1991), 207–220.

[2] X.-Ch. Huang: From Intermediate Value Theorem to chaos. Math. Magazine **65** (1992), 91–103.

[3] H. Lausch and W. Nöbauer: Algebra of polynomials. North Holland, Amsterdam, Amer. Elsevier, New York, 1973.

[4] T.-Y. Li and J.A. Yorke: Period three implies chaos. Amer. Math. Monthly **82** (1975), 985–992.

Helmut Länger
Technische Universität Wien
Institut für Algebra und Diskrete Mathematik
Abteilung für Mathematik in den Naturwissenschaften
und Mathematische Biologie
Wiedner Hauptstraße 8–10
A-1040 Wien