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Some Inequalities for Planar Convex Figures

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Martin Henk was born 1963 in Siegen. He was a student at the University of Siegen where he received his doctoral degree in 1991 and where he now is a scientific assistant. His main subjects are geometry of numbers, convex geometry and optimization.

George A. Tsintsifas was born in 1937 in Thessaloniki, Greece. He obtained his degree in mathematics in 1960 at the University of Thessaloniki. Later he founded a private school and taught geometry at college level. In 1972 he published a problem book on euclidean geometry and geometric transformations. His main subjects are convexity and geometric inequalities.

1 Introduction

Throughout this paper E^2 denotes the 2-dimensional Euclidean space with norm $\|\cdot\|$ and dot product $\langle\cdot,\cdot\rangle$. The set of plane convex figures — compact convex sets — in

Viele der ganz grossen Probleme der Mathematik lassen sich bis weit in die Vergangenheit zurückverfolgen, nicht wenige davon haben sogar Spuren in der klassischen griechischen und lateinischen Literatur hinterlassen. Unter diesen letzteren befindet sich das Problem von Dido, das in Vergils Aeneis angesprochen wird. Dido, die Gründerin und nachmalige Königin Karthagos, stand — mathematisch — vor der Aufgabe, mit einer Kurve gegebener Länge, nämlich der in Streifen geschnittenen Haut eines Stieres, ein Gebiet mit möglichst grossem Flächeninhalt zu umschliessen. Dieses Maximum wird "offensichtlich" dann erreicht, wenn die Kurve eine Kreislinie ist. Aber der *Beweis* für diese *Vermutung* ist bekanntlich nicht so einfach. Jakob und Johann Bernoulli, Leonard Euler auf analytischem und Jakob Steiner auf geometrischem Wege gaben dafür zwar Beweise an. Diese stellten sich aber alle als unvollständig heraus. Ein mathematisch strenger Beweis wurde erstmals in den 70er Jahren des vergangenen Jahrhunderts von Karl Weierstrass mit Hilfe der Variationsrechnung geliefert. Später vervollständigte Wilhelm Blaschke auch Steiners Beweisansatz. Dido's Problem hat zu einer grossen Anzahl von Verallgemeinerungen Anlass gegeben, die heute unter dem Namen *isoperimetrische Probleme* bekannt sind. — Im vorliegende Beitrag von Henk und Tsintsifas geht es um eine Ungleichung aus dem Umfeld des ursprünglichen Problems von Dido: Der Inhalt einer konvexen ebenen Figur wird durch ihren Durchmesser und ihren Inradius abgeschätzt. *ust*

E^2 is denoted by $\mathcal{K}^2$ and the area (perimeter) of $K \in \mathcal{K}^2$ is denoted by $A(K)$ $(L(K))$. The symbols $D(K)$, $\Delta(K)$, $R(K)$, $r(K)$ denote the diameter, width, circumradius and inradius, respectively. These functionals can be defined in the following way (cf. [BF]):

Definition 1.1 *For $c \in E^2$ and $\rho \in \mathbb{R}^{\geq 0}$ let $B(c, \rho) = \{x \in E^2 : \|x - c\| \leq \rho\}$ and let $S^1 = \{x \in E^2 : \|x\| = 1\}$. For $K \in \mathcal{K}^2$ let*

$$\begin{aligned} D(K) &:= \max\{\|x - y\| : x, y \in K\}, \\ \Delta(K) &:= \min_{u \in S^1}\{\max_{x \in K}\langle u, x\rangle - \min_{x \in K}\langle u, x\rangle\}, \\ R(K) &:= \min\{\rho \in \mathbb{R}^{\geq 0} : \exists c \in E^2 \text{ with } K \subset B(c, \rho)\}, \\ r(K) &:= \max\{\rho \in \mathbb{R}^{\geq 0} : \exists c \in E^2 \text{ with } B(c, \rho) \subset K\}. \end{aligned}$$

Thus $D(K)$ is the maximal distance of two points $x, y \in K$, $\Delta(K)$ is the minimal distance of two parallel supporting lines with respect to K, $R(K)$ is the minimal radius of a circle containing K and $r(K)$ is the maximal radius of a circle which is contained in K.

Finally, for a subset $P \subset E^2$ the convex (affine) hull of P is denoted by $\text{conv}(P)$ $(\text{aff}(P))$ and the interior of P is denoted by $\text{int}(P)$.

It is not hard to see that for $K \in \mathcal{K}^2$ the area $A(K)$ is bounded from above and below by the diameter and inradius. Indeed, using the well known inequalities $D(K)\Delta(K) \leq 2A(K) \leq 2D(K)\Delta(K)$ [K] we get immediately the lower bound $A(K) \geq D(K)r(K)$ which in general can not be improved. Applying Blaschke's inequality $\Delta(K) \leq 3r(K)$ [BL] to the upper bound yields $A(K) \leq 3D(K)r(K)$. The purpose of this paper is to prove

Theorem 1.1 *Let $K \in \mathcal{K}^2$. Then*

$$A(K) \leq 2D(K)r(K),$$

and equality holds iff $\text{int}(K) = \emptyset$. *In the case* $\text{int}(K) \neq \emptyset$ *this bound is in general best possible.*

2 Proof of the Theorem

The proof can briefly be described as follows: If $\text{int}(K) = \emptyset$ we have $A(K) = r(K) = 0$ and thus equality. In the other case we consider an affine regular hexagon H inscribed in K with certain properties. From that hexagon we construct another hexagon $\overline{H}$ containing K such that the area of $\overline{H}$ is bounded from above by $2D(K)r(K)$. The proof is prepared by the following Lemma

Lemma 2.1 *Let $P = \text{conv}\{x^1, x^2, x^3, x^4\} \in \mathcal{K}^2$ be a parallelogramme. With the notation of Figure 1 we have for $\|x - y\| \leq (\|x^1 - x^2\|/2)$*

$$A(\text{conv}\{a, x^1, x\}) + A(\text{conv}\{b, x^2, y\}) \geq A(\text{conv}\{x, y, z\}).$$

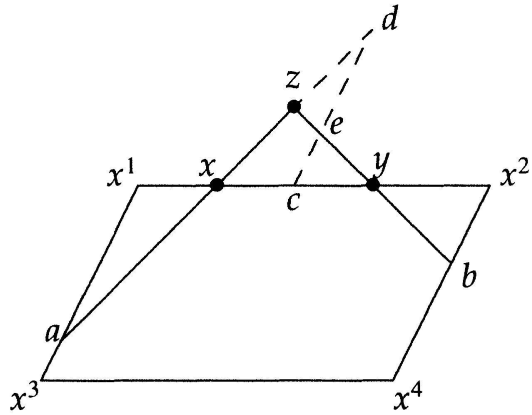

Fig. 1

Proof. Let a, b be arbitrary points in $\operatorname{conv}\{x^1, x^3\}$, $\operatorname{conv}\{x^2, x^4\}$. Without loss of generality let $\|x^1 - x\| \leq \|x^2 - y\|$ and let c be the point in $\operatorname{conv}\{x, x^2\}$ with $\|x - x^1\| = \|x - c\|$. The ray $c + \lambda(x^1 - x^3)$, $\lambda \geq 0$, intersects the ray $a + \mu(z - a)$, $\mu \geq 0$, in a point d and it follows

$$A(\operatorname{conv}\{x, c, d\}) = A(\operatorname{conv}\{a, x^1, x\}). \tag{2.1}$$

In the case $\|c - x\| \geq \|x - y\|$ we have $\operatorname{conv}\{x, y, z\} \subset \operatorname{conv}\{x, c, d\}$, and the claimed inequality is proved in this case. In the other case the ray $c + \lambda(x^1 - x^3)$, $\lambda \geq 0$, also intersects the ray $b + \mu(z - b)$, $\mu \geq 0$, in a point e. We get

$$\operatorname{conv}\{x, y, z\} \subset \operatorname{conv}\{x, c, d\} \cup \operatorname{conv}\{c, y, e\}. \tag{2.2}$$

By assumption we have $\|c - y\| \leq \|x^2 - y\|$ and thus

$$A(\operatorname{conv}\{c, y, e\}) \leq A(\operatorname{conv}\{b, x^2, y\}).$$

On account of (2.1) and (2.2) we obtain the desired inequality. □

Proof of Theorem 1.1. If $\operatorname{int}(K) = \emptyset$ then we have $A(K) = r(K) = 0$ and thus equality. So we may assume $\operatorname{int}(K) \neq \emptyset$. Let $H = \operatorname{conv}\{x^i,\ 1 \leq i \leq 6\}$ be an affine regular hexagon inscribed in K with midpoint 0 (see Figure 2).

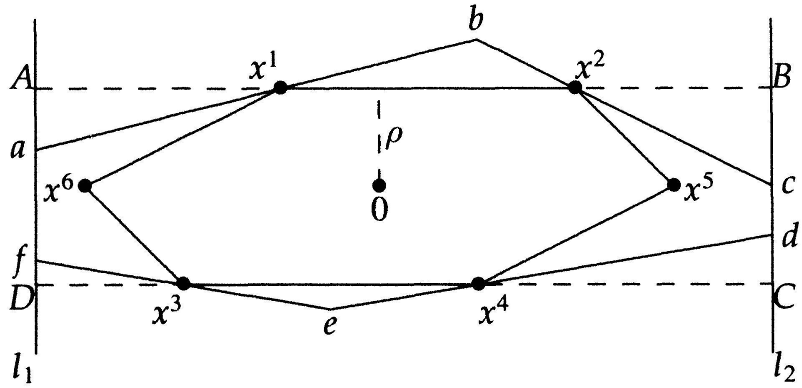

Fig. 2

Let $D(H) = \|x^5 - x^6\|$. Then it is well known ([JB, p. 24,25, pp. 124], [E]) that x^i belongs to the boundary of K, $1 \leq i \leq 6$, $\|x^1 - x^2\| = \|x^3 - x^4\| = \|x^5 - x^6\|/2$ and the edges $\text{conv}\{x^1, x^2\}$, $\text{conv}\{x^3, x^4\}$ have maximal length among the edges of H. Thus the ball with center 0 and radius ρ (distance of $\text{conv}\{x^1, x^2\}$ to 0) is contained in H. Hence

$$\rho = r(H) \leq r(K). \tag{2.3}$$

Let l_1, l_2 be two parallel supporting lines of K with normal vector x^5 and let A, B, C, D denote the intersection points of $\text{aff}\{x^1, x^2\}$, $\text{aff}\{x^3, x^4\}$ with these lines. Then we have

$$\|A - B\| = \|D - C\| \leq D(K). \tag{2.4}$$

Now, let u_i be supporting lines on K through the points x^i, $1 \leq i \leq 4$. The intersection points with the lines l_1, l_2 are denoted by a, c, d, f, respectively. Since x^i, $1 \leq i \leq 4$, belong to the boundary of K we have $l_1 \cap K \cap \text{conv}\{A, D\} \neq \emptyset$ and $l_2 \cap K \cap \text{conv}\{B, C\} \neq \emptyset$. Thus $a, f \in \text{conv}\{A, D\}$ and $c, d \in \text{conv}\{B, C\}$. The intersection point of the lines u^1, u^2 (u^3, u^4) is denoted by b (e). Let $\overline{H} = \text{conv}\{a, b, c, d, e, f\}$. Obviously,

$$K \subset \overline{H} = \text{conv}\{a, x^1, x^2, c, d, x^4, x^3, f\} \cup \text{conv}\{x^1, x^2, b\} \cup \text{conv}\{x^4, e, x^3\}. \tag{2.5}$$

By Lemma 1.1. we get $A(\text{conv}\{x^1, x^2, b\}) \leq A(\text{conv}\{a, A, x^1\}) + A(\text{conv}\{x^2, B, c\})$ and $A(\text{conv}\{x^4, e, x^3, \}) \leq A(\text{conv}\{d, C, x^4\}) + A(\text{conv}\{x^3, D, f\})$ and thus by (2.5), (2.4) and (2.3)

$$A(K) \leq A(\overline{H}) \leq A(\text{conv}\{A, B, C, D\}) = \|A - B\| \cdot 2\rho \leq 2D(K)r(K).$$

To show that this inequality is strict, suppose $r(K) = \rho$. Then two parallel edges of H belong to the boundary of K and each of these edges has a common point with the insphere of radius $r(K)$. Thus K is contained in the parallel strip associated to these edges and hence $A(K) < 2D(K)r(K)$. This shows $A(K) = 2D(K)r(K)$ iff $\text{int}(K) = \emptyset$. Furthermore, the example of the rectangle $Q(q) = \{(x_1, x_2)^T \in E^2 \mid |x_1| \leq q,\ |x_2| \leq 1\}$, $q \in \mathbb{R}^{\geq 0}$ with $q \to \infty$ shows that in general this inequality can not be improved in the case $\text{int}(K) \neq \emptyset$. □

3 Further inequalities

In this section we collect some inequalities for plane convex figures which are closely related to Theorem 1.1.

(1) $(\sqrt{3}/2)\Delta(K)R(K) \leq A(K) \leq 2\Delta(K)R(K)$.
For the lower bound see [He, p. 29], the upper bound can be easily deduced from $A(K) \leq \Delta(K)D(K)$ [K].

(2) $L(K)r(K) \leq 2A(K) \leq 2r(K)(L(K) - \pi r(K))$.
The upper bound is due to Bonnesen [Bo] and the obvious lower bound can be found in [BF, p. 82].

(3) $4R(K) \leq L(K) \leq 2D(K) + 4r(K)$.
For the lower bound see [N], [CK]. The upper bound follows by Theorem 1.1. and the well known Favard's inequality $L(K)D(K) \leq 2A(K) + 2D(K)^2$ (cf. [F], [RR]).

(4) $2R(K)r(K) \leq A(K) \leq 4R(K)r(K)$.
The lower bound follows from the obvious inequality $2A(K) \geq L(K)r(K)$ [BF] and the lower bound in (3). The upper bound is an immediate consequence of Theorem 1.1 and $D(K) \leq 2R(K)$.

(5) $R(K)(L(K) - 4R(K)) \leq A(K) \leq 2R(K)(L(K) - 2R(K))$.
The lower bound is due to Favard [F]. For the upper bound let 0 be the center of an insphere with radius $r(K)$. Then it is easy to see that K is contained in the ball with radius $D(K) - r(K)$ and center 0. Thus $D(K) \geq R(K) + r(K)$ and by $2D(K) \leq L(K)$ we get $L(K) - 2R(K) \geq 2r(K)$. Together with the upper bound in (4) we obtain the desired inequality.

(6) $D(K)(L(K) - 2D(K)) \leq 2A(K) \leq L(K)D(K)/2$.
The lower bound is also due to Favard [F] and the upper bound is due to Hayashi [H].

References

[Bl] W. Blaschke: *Über den größten Kreis in einer konvexen Punktmenge*, Jber. Deutsch. Math. Vereinig. **23** (1914), 369–374.

[Bo] T. Bonnesen: *Les problèmes des isopérimètres et des isépiphanes*, Paris, 1929.

[BF] T. Bonnesen, W. Fenchel: *Theorie der konvexen Körper*, Springer, Berlin, 1934.

[CK] G.D. Chakerian, M.S. Klamkin: *Minimal covers for closed curves*, Math. Magazine **78(2)** (1973), 55–61.

[E] H.G. Eggleston: *Convexity*, Cambridge Univ. Press, Cambridge, 1958.

[F] J. Favard: *Problèmes d'extremums relatifs aux courbes convexes I. II.*, Ann. Ecole norm. **46** (1929), 345–369.

[H] T. Hayashi: *The extremal chords of an oval*, Tôhoku Math. J. **22** (1923), 387–393.

[He] M. Henk: *Ungleichungen für sukzessive Minima und verallgemeinerte In- und Umkugelradien*, PhD thesis, University of Siegen, 1991.

[JB] I.M. Jaglom, W.G. Boltjanski: *Konvexe Figuren*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956.

[K] T. Kubota: *Einige Ungleichheitsbeziehungen über Eilinien und Eiflächen*, Sci. Rep. Tôhoku Univ. **12** (1923), 45–65.

[N] J.C.C. Nitsche: *The smallest sphere containing a rectifiable curve*, Am. Math. Monthly **87** (1971), 881–882.

[RR] M. Rogialski, J.S. Raymond: *Inequalities about symmetric compact convex sets in the plane*, Am. Math. Monthly **92(7)** (1985), 466–480.

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Die Literatur über isoperimetrische Probleme füllt Bibliotheken. Als Einführung in das ursprüngliche geometrische Problem in zwei und drei Dimensionen sowie in die anschliessenden Fragen über konvexe Gebiete ist das alte Buch

W. Blaschke: *Kreis und Kugel.* Verlag von Veit & Co. Leipzig (1916).

immer noch lesenswert. Blaschke hat ferner seiner "Antrittsrede" vom 15. Mai 1915 an der Universität Leipzig den gleichen Titel gegeben. Dieser für ein allgemeineres Publikum gehaltene Vortrag ist zugänglich in

C. Neumann, et al.: *Leipziger mathematische Antrittsvorlesungen.* Teubner Archiv zur Mathematik, Teubner Verlagsgesellschaft, Leipzig (1987).

Von den mathematisch anspruchsvollen Darstellungen des Gebietes erwähnen wir neben dem in der obigen Literaturliste aufgeführten klassischen Werk von T. Bonnesen und W. Fenchel die beiden folgenden Bücher:

C. Bandle: *Isoperimetric inequalities and applications.* Monographs and Studies in Mathematics 7, Pitman (1980).

Yu.D. Burago, V.A. Zalgaller: *Geometric inequalities.* Grundlehren der mathematischen Wissenschaften 285, Springer Verlag (1988).