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A Good Basis for Computing with Complex Numbers

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Alain Robert, né en 1941, a obtenu son doctorat en 1967. Après des séjours à Paris et Princeton, il a été nommé professeur à l'université de Neuchâtel en 1971 où il enseigne l'analyse. Il a aussi enseigné à Kingston (Canada), Rio-de-Janeiro et Berkeley. Il est auteur de divers livres dont "Courbes elliptiques", "Représentations des groupes", "Analyse non standard". Il s'intéresse actuellement aux relations entre nombres p-adiques et fractals.

Das Dezimalsystem beginnt mit den natürlichen Zahlen: Es codiert sie als Wörter endlicher Länge über dem Alphabet $\{0,\ldots,9\}$. Mit Hilfe eines Vorzeichens lassen sich dann auch beliebige ganze Zahlen in eindeutiger Weise darstellen. Die ganzen Zahlen $\mathbb{Z}$ bilden im System $\mathbb{R}$ der reellen Zahlen einerseits einen Ring und anderseits ein Gitter mit dem Schrittintervall $[0,1] =: F$ als Fundamentalbereich. Jede reelle Zahl $\alpha \in F$ wird vom Dezimalsystem als unendliche Nachkommazahl, Beispiel: .26391145..., codiert, wobei die sinngemäss interpretierten endlichen Anfangsstücke dieses Wortes die gemeinte Zahl α besser und besser approximieren. Und noch etwas: Die Menge der höchstens n-stelligen natürlichen Zahlen ist eine (um den Faktor 10^n vergrösserte) "gerasterte" Kopie von F.

Schon verschiedentlich ist der Versuch gemacht worden, eine analoge Basisdarstellung für die *komplexen* Zahlen zu komponieren. Dazu benötigt man eine geeignete Basis $b \in \mathbb{C}$ und ein zugehöriges Alphabet S derart, dass jede Zahl $\zeta \in \mathbb{C}$ ohne Rückgriff auf Real- und Imaginärteil mehr oder weniger eindeutig als Summe $\sum_{k\leq r} a_k b^k$, die $a_k \in S$, darstellbar ist und damit als Wort $a_r a_{r-1} \ldots a_1 a_0 . a_{-1} a_{-2} \ldots$ codiert werden kann. Der hier von Robert unterbreitete Vorschlag ist darum besonders reizvoll, weil der entstehende Fundamentalbereich F nicht etwa ein Gitterparallelogramm ist, sondern eine fraktale Grenze aufweist. Die oben für $\mathbb{Z}$ und $\mathbb{R}$ beschriebenen Sachverhalte bleiben gültig; insbesondere gleicht die Menge der höchstens n-stelligen "ganzen" Zahlen für wachsendes n immer mehr einer stark vergrösserten Kopie von F.

Aufgabe für den Leser: Ein Rechenbuch für das Umgehen mit derartigen komplexen b-Brüchen zu verfassen, beginnend mit dem "kleinen Einmaleins" und endend mit Regeln für das "Schriftlichrechnen". *cbl*

Several bases for computing with complex numbers have been studied. We propose one which gives a numbering system for the ring of integers L in $\mathbb{Q}(\sqrt{-3})$. This ring of integers is generated by $\zeta = (1+\sqrt{-3})/2 = \exp(\pi i/3)$ (a primitive sixth root of 1) and is a hexagonal lattice in $\mathbb{C}$.

We propose to take the basis $b = \sqrt{-3} = i\sqrt{3}$, and we let the corresponding digits be $S = \{0, 1, \zeta\}$. Our purpose thus is to consider finite sums

$$a_0 + a_1\sqrt{-3} + a_2(-3) + a_3(-3\sqrt{-3}) + \cdots + a_n(\sqrt{-3})^n$$

with digits $a_i \in S$.

In Figure 1 we picture the first nine elements of this system: 0, $\sqrt{-3}$, $\zeta\sqrt{-3}$, 1, $1+\sqrt{-3}$, $1+\zeta\sqrt{-3}$, ζ, $\zeta+\sqrt{-3}$, $\zeta+\zeta\sqrt{-3}$.

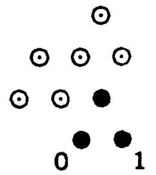

Fig. 1

Figure 2 shows the growth of the basic scheme with the first 81 and $3 \cdot 81$ points. Let $S^{\mathbb{N}}$ denote the set of families $(a_i)_{i \geq 0}$ of elements of S with $a_i \neq 0$ for finitely many indices only (i.e. $a_i = 0$ for $i \gg 0$).

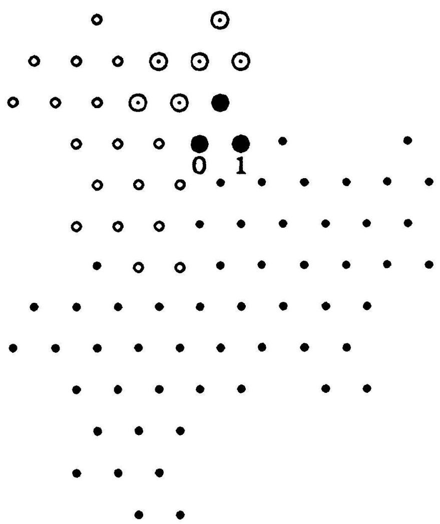

Fig. 2

Proposition 1. *The mapping* $\Phi : S^{\mathbb{N}} \rightarrow \mathbb{C}$, $(a_i) \mapsto \sum a_i b^i$ *is injective with image* $L = \mathbb{Z} \oplus \mathbb{Z}\zeta = \mathbb{Z}[\zeta] \subset \mathbb{Q}(\sqrt{-3})$.

Proof. Since L is a ring containing b and S, the image of Φ is in L. On the other hand, $L = \mathbb{Z} \oplus \mathbb{Z}\zeta$ is the free abelian group generated by 1 and ζ. It will be enough to show that the image of Φ, namely the set of finite sums $\sum a_i b^i$ $(a_i \in S)$ is also a group (hence a subgroup of L containing 1 and ζ). For this purpose we have to show how to bring sums and differences of elements in the image of Φ in reduced form. It is enough to give reduced expressions for the elements of $-S$ and of $S+S$. But from $\zeta = (1+b)/2$ we have $2\zeta = 1 + b$, and

$$-1 = b - 2\zeta = \zeta + b - 3\zeta = \zeta + b + \zeta b^2$$

gives

$$-\zeta = 1 + b + \zeta b^2 .$$

Similarly one proves

$$2 = \zeta + b + b^2 + \zeta b^3 , \quad 1 + \zeta = \zeta b + \zeta b^2 + b^3 + \zeta b^4 .$$

(Although we do not need it later, let us also note $\zeta^2 = 1 + \zeta b$. This shows that $\operatorname{Im} \Phi$ is a subring.) We still have to prove that the mapping Φ is *injective*. This is easily seen by using the field $\mathbb{Q}_3$ of 3-adic numbers and its quadratic extension K obtained by adjoining a square root of -3. One can think of $\mathbb{Q}_3$ as consisting of formal (infinite) expansions

$$\sum_{i \geq k} a_i 3^i \quad (a_i \in \{0,1,2\},\ k \in \mathbb{Z}) .$$

(One can also work with representatives $a_i = 0$ or ± 1, or any other set S of representatives of $\mathbb{Z}$ mod $3\mathbb{Z}$.) The sums $\sum_{i \geq 0} a_i 3^i$ make up the subring $\mathbb{Z}_3$ of 3-adic integers: this is a maximal subring in $\mathbb{Q}_3$. The ring $\mathbb{Z}_3$ is a principal ideal domain; its ideals are of the form $3^l \mathbb{Z}_3$ $(l \geq 0)$. The quadratic extension $K = \mathbb{Q}_3(b)$ where $b^2 = -3$ also contains a maximal subring R; it is not hard to see that R too is a principal ideal domain and that its ideals are of the form $b^l R$ $(l \geq 0)$. In particular, it has a unique maximal ideal $P = bR$, the corresponding quotient is

$$R/P = \mathbb{Z}_3/3\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z} = \mathbb{F}_3$$

where $\mathbb{F}_3$ denotes the field with 3 elements. (One says that the quadratic extension $K/\mathbb{Q}_3$ is *totally ramified*.) If a set of representatives S of R mod P is chosen, then the elements of K admit unique expansions

$$\sum_{i \geq k} a_i b^i \quad (a_i \in S,\ k \in \mathbb{Z}) ,$$

and the elements of R admit unique expansions

$$\sum_{i \geq 0} a_i b^i \quad (a_i \in S) .$$

I claim that we can take $S = \{0, 1, \zeta\}$. Indeed, we have seen

$$-1 = \zeta + b + \zeta b^2$$

and this proves that $\zeta \equiv -1$ mod P (recall $P = bR$). This already proves that distinct finite sums $\sum_{i \geq 0} a_i b^i$ correspond to distinct elements of L (they are distinct in R!) $\square$

Let us now consider the index set $I = \{i < 0,\ i \in \mathbb{Z}\}$ and the compact set S^I (with the product of the discrete topologies). Since $|b^{-l}| < 1$ the series

$$\sum_{i<0} a_i b^i$$

converge absolutely in $\mathbb{C}$ for all sequences $a = (a_i) \in S^I$, and we obtain a continuous map

$$S^I \to \mathbb{C} :\ a = (a_i) \mapsto \sum_{i<0} a_i b^i$$

which we still denote by Φ. Its image $\Phi(S^I) = F$ is a compact subset of $\mathbb{C}$. By definition bF consists of the sums

$$\sum_{i\leq 0} a_i b^i = a_0 + \sum_{i<0} a_i b^i \quad (a_i \in S)$$

so that $bF = F \cup (1 + F) \cup (\zeta + F)$. Similarly, since $b^4 = 9$, we have $9F = \bigcup (a + F)$ where the union is taken aver $a \in S + Sb + Sb^2 + Sb^3$. The homothetic $9F$ of F is made up of 81 pieces congruent to F (a puzzle!). Let us denote by $F_k \subset F$ the finite part consisting of the sums $\sum_{-k\leq i<0} a_i b^i$. It is obvious that $F_\infty = \bigcup_{k\geq 1} F_k$ is dense in F.

Lemma. *The origin is an interior point of F.*

Proof. It is easy to make pictures of the sets F_k for some small values of k. For example $9F_4 = S + Sb + Sb^2 + Sb^3$ which contains 81 elements is pictured in Figure 2. These pictures show that one can find an open neighbourhood of the origin $U = \{z \mid |z| < \epsilon\}$ in $\mathbb{C}$ such that $F_\infty \cap U$ is dense in U. Hence $F \cap U = U$ since it is both closed and dense in U. (From the fact that the origin has six neighbours in F_4 the assiduous reader will infer an explicit value for ϵ above!)

Proposition 2. (a) *The set F_∞ is contained in the interior of F.*
(b) *The set F_∞ is a set of representatives for $L[1/b]$ mod L, i.e. $L[1/b]$ can be written as disjoint union:*

$$L[1/b] = \bigcup_{\gamma\in L} (\gamma + F_\infty) .$$

Proof. To prove (a), observe that if $z \in F_\infty$, say $z = \sum_{-k\leq i<0} a_i b^i \in F_k$, then $z + b^{-k}F \subset F$. Hence z is an interior point of F by the lemma. For (b) observe that an element z of $b^{-k}L$ has an expression as a *finite* sum $\sum_{-k\leq i} a_i b^i$ and accordingly can be decomposed

$$z = \sum_{-k\leq i<0} a_i b^i + \sum_{i\geq 0} a_i b^i \in \gamma + F_k \quad \text{with} \quad \gamma = \sum_{i\geq 0} a_i b^i \in L .$$

In fact, we see that

$$b^k L = \bigcup_{\gamma\in L} (\gamma + F_k) .$$

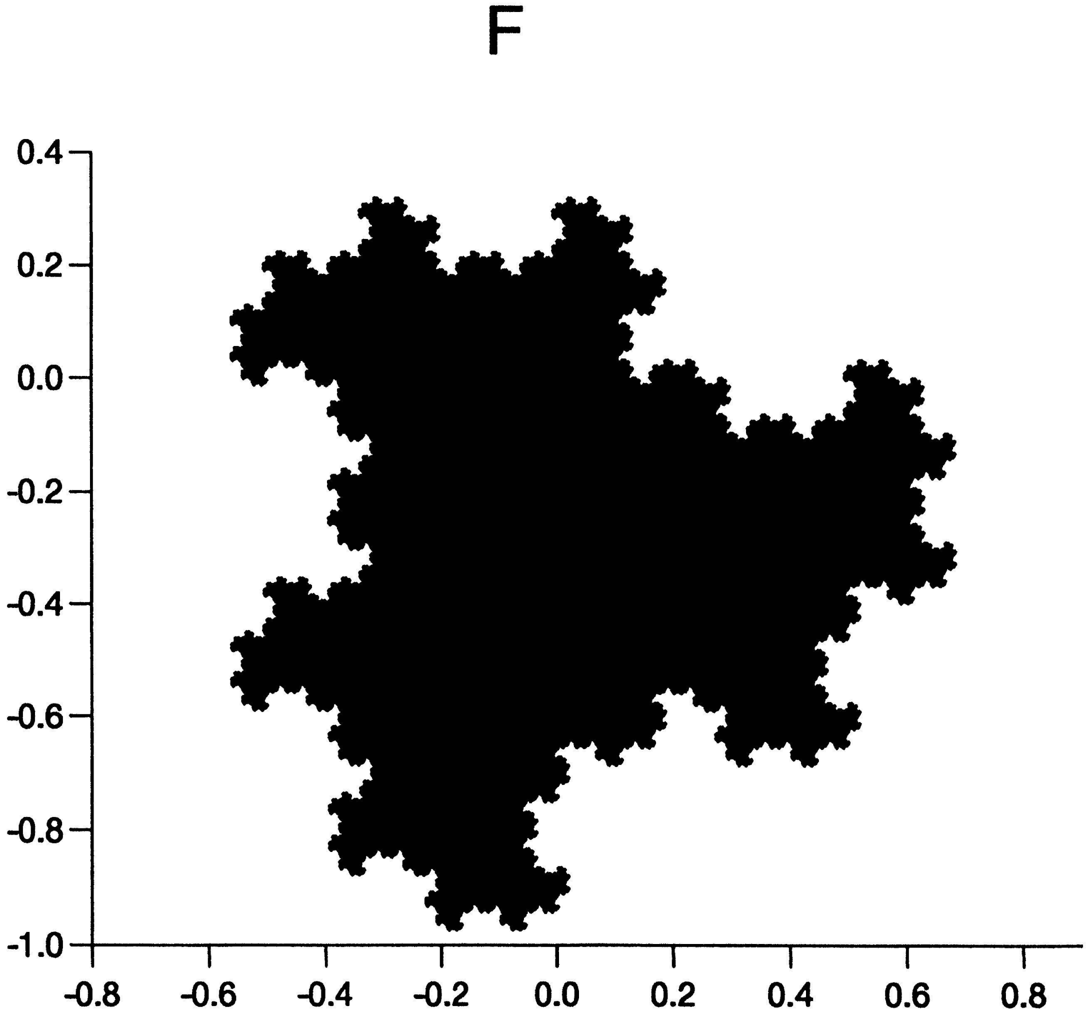

Fig. 3 The set F (Computer image by C. Beguin).

Corollary. *The set F is a fundamental domain for L in* $\mathbb{C}$*:* $\mathbb{C} = F + L$ *and the intersection of two distinct translates of F has no interior point.*

Proof. From the fact that the origin is an interior point of F we deduce that for any $z \in \mathbb{C}$ there is a positive integer k such that $b^{-k}z \in F$. Hence

$$b^{-k}z = \sum_{i<0} a_i b^i \quad (a_i \in S) \ ,$$

$$z = \sum_{i<0} a_i b^{i+k} = \sum_{j<0} a_{j-k} b^j + \sum_{0 \leq j < k} a_{j-k} b^j \in F + L \ .$$

This proves that $\mathbb{C} = \bigcap_{\gamma \in L}(\gamma + F)$. On the other hand, if $F \cap (\gamma + F)$ has an interior point, let B be an open disc (of positive radius) contained in this intersection. Then

$$B \cap F_\infty \subset (\gamma + F) \cap L[1/b] = \gamma + F_\infty$$

proves that $(\gamma + F_\infty) \cap F_\infty$ is not empty.

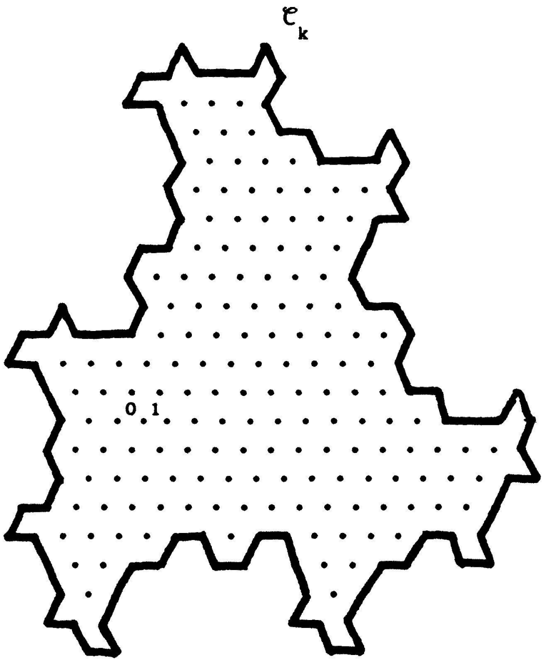

Fig. 4

Proposition 3. *The boundary of F is a Jordan curve $\mathcal{C}$ of fractal dimension $d = \log 4/\log 3 = 1,26\ldots$*

Proof. Coming back to the pictures of the sets F_k for small values of k, e.g. to F_4 or its homothetic $9F = S + Sb + Sb^2 + Sb^3$, containing 81 elements, we see that we can take polygonal lines $\mathcal{C}_k$ with vertices at the boundary points of F_k (points having less than 6 neighbours in F_k, cf. Figure 4). The curves $\mathcal{C}_k$ converge uniformly to a curve $\mathcal{C}$. To determine the dimension of $\mathcal{C}$ we observe that $\sqrt{-3}\cdot\mathcal{C}$ is a union of two copies of $\mathcal{C}$: Figure 5 shows how two copies of $\mathcal{C}$ are used to reconstruct this homothetic

$$\sqrt{-3}\cdot\mathcal{C} = \sqrt{-3}\cdot\partial F = \partial(\sqrt{-3}\cdot F)\ .$$

Since the similarity dimension d of a set A is defined by

$$\mathrm{Size}(\lambda A) = \lambda^d\cdot\mathrm{Size}(A)$$

we see that, in our case, we must have $(\sqrt{3})^d = 2$, whence $d\cdot\log\sqrt{3} = \log 2$. This proves Proposition 3.

Corollary. *The set F is connected (it is homeomorphic to a closed disc). The interior of F is the bounded component of $\mathbb{C} - \mathcal{C}$ and F is equal to the closure of its interior.*

Observe that the dimension of $\mathcal{C}$ is the same as the dimension of the von Koch curve.

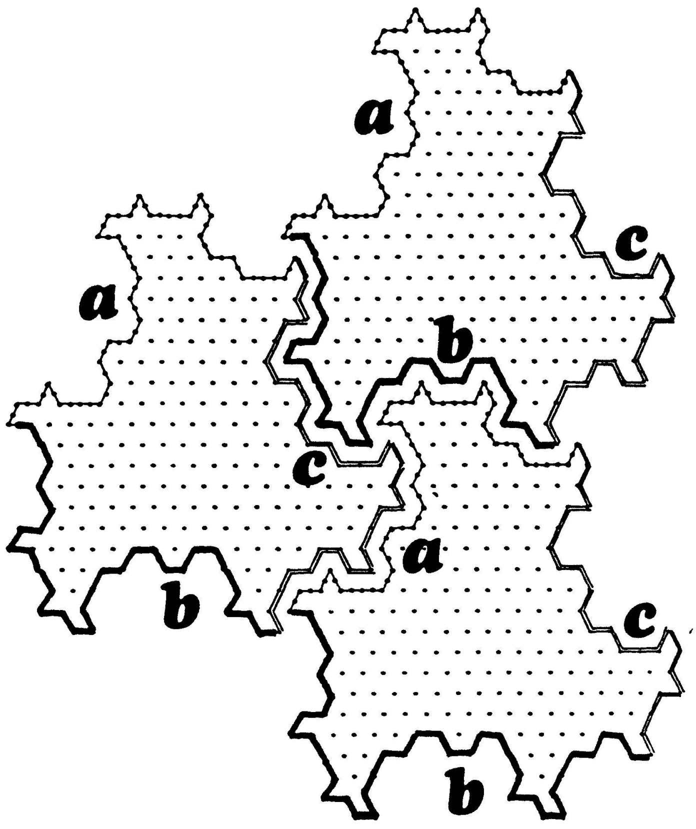

Fig. 5

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