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## A Good Basis for Computing with Complex Numbers

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Alain Robert

Alain Robert, né en 1941, a obtenu son doctorat en 1967. Après des séjours à Paris et Princeton, il a été nommé professeur à l'université de Neuchâtel en 1971 où il enseigne l'analyse. Il a aussi enseigné à Kingston (Canada), Rio-de-Janeiro et Berkeley. Il est auteur de divers livres dont "Courbes elliptiques", "Représentations des groupes", "Analyse non standard". Il s'intéresse actuellement aux relations entre nombres  $p$ -adiques et fractals.

Das Dezimalsystem beginnt mit den natürlichen Zahlen: Es codiert sie als Wörter endlicher Länge über dem Alphabet  $\{0, \dots, 9\}$ . Mit Hilfe eines Vorzeichens lassen sich dann auch beliebige ganze Zahlen in eindeutiger Weise darstellen. Die ganzen Zahlen  $\mathbb{Z}$  bilden im System  $\mathbb{R}$  der reellen Zahlen einerseits einen Ring und andererseits ein Gitter mit dem Schrittintervall  $[0, 1] =: F$  als Fundamentalbereich. Jede reelle Zahl  $\alpha \in F$  wird vom Dezimalsystem als unendliche Nachkommazahl, Beispiel: .26391145... codiert, wobei die sinngemäss interpretierten endlichen Anfangsstücke dieses Wortes die gemeinte Zahl  $\alpha$  besser und besser approximieren. Und noch etwas: Die Menge der höchstens  $n$ -stelligen natürlichen Zahlen ist eine (um den Faktor  $10^n$  vergrößerte) "gerasterte" Kopie von  $F$ .

Schon verschiedentlich ist der Versuch gemacht worden, eine analoge Basisdarstellung für die komplexen Zahlen zu komponieren. Dazu benötigt man eine geeignete Basis  $b \in \mathbb{C}$  und ein zugehöriges Alphabet  $S$  derart, dass jede Zahl  $\zeta \in \mathbb{C}$  ohne Rückgriff auf Real- und Imaginärteil mehr oder weniger eindeutig als Summe  $\sum_{k \leq r} a_k b^k$ , die  $a_k \in S$ , darstellbar ist und damit als Wort  $a_r a_{r-1} \dots a_1 a_0 . a_{-1} a_{-2} \dots$  codiert werden kann. Der hier von Robert unterbreitete Vorschlag ist darum besonders reizvoll, weil der entstehende Fundamentalbereich  $F$  nicht etwa ein Gitterparallelogramm ist, sondern eine fraktale Grenze aufweist. Die oben für  $\mathbb{Z}$  und  $\mathbb{R}$  beschriebenen Sachverhalte bleiben gültig; insbesondere gleicht die Menge der höchstens  $n$ -stelligen "ganzen" Zahlen für wachsendes  $n$  immer mehr einer stark vergrößerten Kopie von  $F$ .

**Aufgabe für den Leser:** Ein Rechenbuch für das Umgehen mit derartigen komplexen  $b$ -Brüchen zu verfassen, beginnend mit dem "kleinen Einmaleins" und endend mit Regeln für das "Schriftlichrechnen". *cbl*

Several bases for computing with complex numbers have been studied. We propose one which gives a numbering system for the ring of integers  $L$  in  $\mathbb{Q}(\sqrt{-3})$ . This ring of integers is generated by  $\zeta = (1 + \sqrt{-3})/2 = \exp(\pi i/3)$  (a primitive sixth root of 1) and is a hexagonal lattice in  $\mathbb{C}$ .

We propose to take the basis  $b = \sqrt{-3} = i\sqrt{3}$ , and we let the corresponding digits be  $S = \{0, 1, \zeta\}$ . Our purpose thus is to consider finite sums

$$a_0 + a_1\sqrt{-3} + a_2(-3) + a_3(-3\sqrt{-3}) + \dots + a_n(\sqrt{-3})^n$$

with digits  $a_i \in S$ .

In Figure 1 we picture the first nine elements of this system:  $0, \sqrt{-3}, \zeta\sqrt{-3}, 1, 1 + \sqrt{-3}, 1 + \zeta\sqrt{-3}, \zeta, \zeta + \sqrt{-3}, \zeta + \zeta\sqrt{-3}$ .

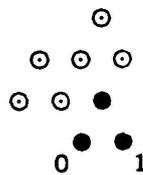


Fig. 1

Figure 2 shows the growth of the basic scheme with the first 81 and  $3 \cdot 81$  points. Let  $S^{\mathbb{N}}$  denote the set of families  $(a_i)_{i \geq 0}$  of elements of  $S$  with  $a_i \neq 0$  for finitely many indices only (i.e.  $a_i = 0$  for  $i \gg 0$ ).

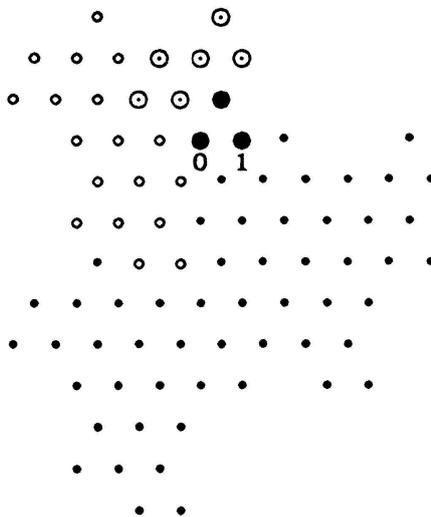


Fig. 2

**Proposition 1.** *The mapping  $\Phi : S^{\mathbb{N}} \rightarrow \mathbb{C}, (a_i) \mapsto \sum a_i b^i$  is injective with image  $L = \mathbb{Z} \oplus \mathbb{Z}\zeta = \mathbb{Z}[\zeta] \subset \mathbb{Q}(\sqrt{-3})$ .*

*Proof.* Since  $L$  is a ring containing  $b$  and  $S$ , the image of  $\Phi$  is in  $L$ . On the other hand,  $L = \mathbb{Z} \oplus \mathbb{Z}\zeta$  is the free abelian group generated by 1 and  $\zeta$ . It will be enough to show that the image of  $\Phi$ , namely the set of finite sums  $\sum a_i b^i$  ( $a_i \in S$ ) is also a group (hence a subgroup of  $L$  containing 1 and  $\zeta$ ). For this purpose we have to show how to bring

sums and differences of elements in the image of  $\Phi$  in reduced form. It is enough to give reduced expressions for the elements of  $-S$  and of  $S + S$ . But from  $\zeta = (1 + b)/2$  we have  $2\zeta = 1 + b$ , and

$$-1 = b - 2\zeta = \zeta + b - 3\zeta = \zeta + b + \zeta b^2$$

gives

$$-\zeta = 1 + b + \zeta b^2 .$$

Similarly one proves

$$2 = \zeta + b + b^2 + \zeta b^3 , \quad 1 + \zeta = \zeta b + \zeta b^2 + b^3 + \zeta b^4 .$$

(Although we do not need it later, let us also note  $\zeta^2 = 1 + \zeta b$ . This shows that  $\text{Im } \Phi$  is a subring.) We still have to prove that the mapping  $\Phi$  is *injective*. This is easily seen by using the field  $\mathbb{Q}_3$  of 3-adic numbers and its quadratic extension  $K$  obtained by adjoining a square root of  $-3$ . One can think of  $\mathbb{Q}_3$  as consisting of formal (infinite) expansions

$$\sum_{i \geq k} a_i 3^i \quad (a_i \in \{0, 1, 2\}, k \in \mathbb{Z}) .$$

(One can also work with representatives  $a_i = 0$  or  $\pm 1$ , or any other set  $S$  of representatives of  $\mathbb{Z} \bmod 3\mathbb{Z}$ .) The sums  $\sum_{i \geq 0} a_i 3^i$  make up the subring  $\mathbb{Z}_3$  of 3-adic integers: this is a maximal subring in  $\mathbb{Q}_3$ . The ring  $\mathbb{Z}_3$  is a principal ideal domain; its ideals are of the form  $3^l \mathbb{Z}_3$  ( $l \geq 0$ ). The quadratic extension  $K = \mathbb{Q}_3(b)$  where  $b^2 = -3$  also contains a maximal subring  $R$ ; it is not hard to see that  $R$  too is a principal ideal domain and that its ideals are of the form  $b^l R$  ( $l \geq 0$ ). In particular, it has a unique maximal ideal  $P = bR$ , the corresponding quotient is

$$R/P = \mathbb{Z}_3/3\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z} = \mathbb{F}_3$$

where  $\mathbb{F}_3$  denotes the field with 3 elements. (One says that the quadratic extension  $K/\mathbb{Q}_3$  is *totally ramified*.) If a set of representatives  $S$  of  $R \bmod P$  is chosen, then the elements of  $K$  admit unique expansions

$$\sum_{i \geq k} a_i b^i \quad (a_i \in S, k \in \mathbb{Z}) ,$$

and the elements of  $R$  admit unique expansions

$$\sum_{i \geq 0} a_i b^i \quad (a_i \in S) .$$

I claim that we can take  $S = \{0, 1, \zeta\}$ . Indeed, we have seen

$$-1 = \zeta + b + \zeta b^2$$

and this proves that  $\zeta \equiv -1 \pmod{P}$  (recall  $P = bR$ ). This already proves that distinct finite sums  $\sum_{i \geq 0} a_i b^i$  correspond to distinct elements of  $L$  (they are distinct in  $R$ !)  $\square$

Let us now consider the index set  $I = \{i < 0, i \in \mathbb{Z}\}$  and the compact set  $S^I$  (with the product of the discrete topologies). Since  $|b^{-1}| < 1$  the series

$$\sum_{i < 0} a_i b^i$$

converge absolutely in  $\mathbb{C}$  for all sequences  $a = (a_i) \in S^I$ , and we obtain a continuous map

$$S^I \rightarrow \mathbb{C} : a = (a_i) \mapsto \sum_{i < 0} a_i b^i$$

which we still denote by  $\Phi$ . Its image  $\Phi(S^I) = F$  is a compact subset of  $\mathbb{C}$ . By definition  $bF$  consists of the sums

$$\sum_{i \leq 0} a_i b^i = a_0 + \sum_{i < 0} a_i b^i \quad (a_i \in S)$$

so that  $bF = F \cup (1 + F) \cup (\zeta + F)$ . Similarly, since  $b^4 = 9$ , we have  $9F = \bigcup (a + F)$  where the union is taken over  $a \in S + Sb + Sb^2 + Sb^3$ . The homothetic  $9F$  of  $F$  is made up of 81 pieces congruent to  $F$  (a puzzle!). Let us denote by  $F_k \subset F$  the finite part consisting of the sums  $\sum_{-k \leq i < 0} a_i b^i$ . It is obvious that  $F_\infty = \bigcup_{k \geq 1} F_k$  is dense in  $F$ .

**Lemma.** *The origin is an interior point of  $F$ .*

*Proof.* It is easy to make pictures of the sets  $F_k$  for some small values of  $k$ . For example  $9F_4 = S + Sb + Sb^2 + Sb^3$  which contains 81 elements is pictured in Figure 2. These pictures show that one can find an open neighbourhood of the origin  $U = \{z \mid |z| < \epsilon\}$  in  $\mathbb{C}$  such that  $F_\infty \cap U$  is dense in  $U$ . Hence  $F \cap U = U$  since it is both closed and dense in  $U$ . (From the fact that the origin has six neighbours in  $F_4$  the assiduous reader will infer an explicit value for  $\epsilon$  above!)

**Proposition 2.** (a) *The set  $F_\infty$  is contained in the interior of  $F$ .*

(b) *The set  $F_\infty$  is a set of representatives for  $L[1/b] \bmod L$ , i.e.  $L[1/b]$  can be written as disjoint union:*

$$L[1/b] = \bigcup_{\gamma \in L} (\gamma + F_\infty).$$

*Proof.* To prove (a), observe that if  $z \in F_\infty$ , say  $z = \sum_{-k \leq i < 0} a_i b^i \in F_k$ , then  $z + b^{-k}F \subset F$ . Hence  $z$  is an interior point of  $F$  by the lemma. For (b) observe that an element  $z$  of  $b^{-k}L$  has an expression as a *finite* sum  $\sum_{-k \leq i} a_i b^i$  and accordingly can be decomposed

$$z = \sum_{-k \leq i < 0} a_i b^i + \sum_{i \geq 0} a_i b^i \in \gamma + F_k \quad \text{with} \quad \gamma = \sum_{i \geq 0} a_i b^i \in L.$$

In fact, we see that

$$b^k L = \bigcup_{\gamma \in L} (\gamma + F_k).$$

# F

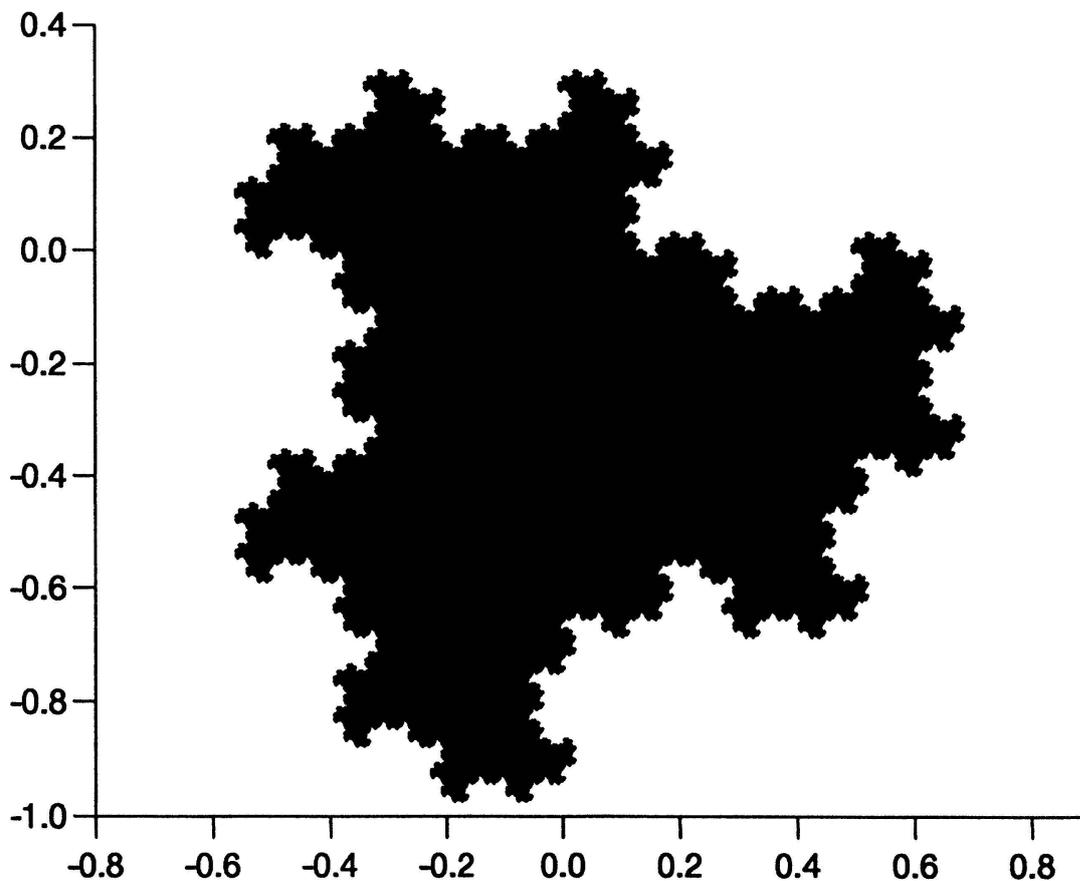


Fig. 3 The set  $F$  (Computer image by C. Begum).

**Corollary.** *The set  $F$  is a fundamental domain for  $L$  in  $\mathbb{C}$ :  $\mathbb{C} = F + L$  and the intersection of two distinct translates of  $F$  has no interior point.*

*Proof.* From the fact that the origin is an interior point of  $F$  we deduce that for any  $z \in \mathbb{C}$  there is a positive integer  $k$  such that  $b^{-k}z \in F$ . Hence

$$b^{-k}z = \sum_{i < 0} a_i b^i \quad (a_i \in S) ,$$

$$z = \sum_{i < 0} a_i b^{i+k} = \sum_{j < 0} a_{j-k} b^j + \sum_{0 \leq j < k} a_{j-k} b^j \in F + L .$$

This proves that  $\mathbb{C} = \bigcup_{\gamma \in L} (\gamma + F)$ . On the other hand, if  $F \cap (\gamma + F)$  has an interior point, let  $B$  be an open disc (of positive radius) contained in this intersection. Then

$$B \cap F_\infty \subset (\gamma + F) \cap L[1/b] = \gamma + F_\infty$$

proves that  $(\gamma + F_\infty) \cap F_\infty$  is not empty.

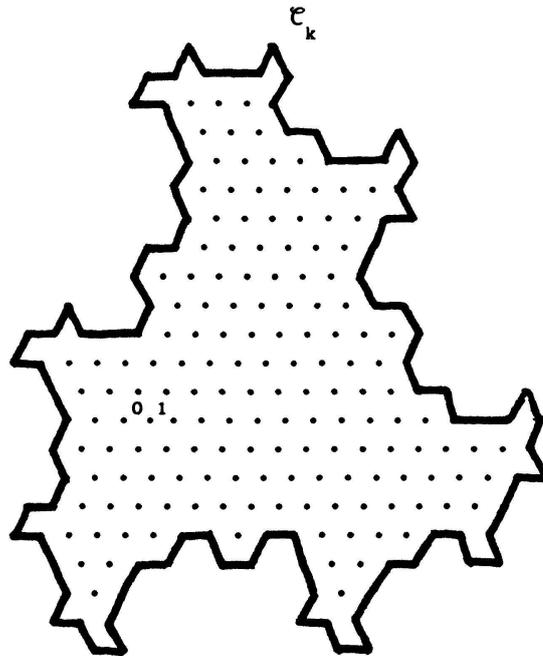


Fig. 4

**Proposition 3.** *The boundary of  $F$  is a Jordan curve  $\mathcal{C}$  of fractal dimension  $d = \log 4 / \log 3 = 1,26\dots$*

*Proof.* Coming back to the pictures of the sets  $F_k$  for small values of  $k$ , e.g. to  $F_4$  or its homothetic  $9F = S + Sb + Sb^2 + Sb^3$ , containing 81 elements, we see that we can take polygonal lines  $\mathcal{C}_k$  with vertices at the boundary points of  $F_k$  (points having less than 6 neighbours in  $F_k$ , cf. Figure 4). The curves  $\mathcal{C}_k$  converge uniformly to a curve  $\mathcal{C}$ . To determine the dimension of  $\mathcal{C}$  we observe that  $\sqrt{-3} \cdot \mathcal{C}$  is a union of two copies of  $\mathcal{C}$ : Figure 5 shows how two copies of  $\mathcal{C}$  are used to reconstruct this homothetic

$$\sqrt{-3} \cdot \mathcal{C} = \sqrt{-3} \cdot \partial F = \partial(\sqrt{-3} \cdot F) .$$

Since the similarity dimension  $d$  of a set  $A$  is defined by

$$\text{Size}(\lambda A) = \lambda^d \cdot \text{Size}(A)$$

we see that, in our case, we must have  $(\sqrt{3})^d = 2$ , whence  $d \cdot \log \sqrt{3} = \log 2$ . This proves Proposition 3.

**Corollary.** *The set  $F$  is connected (it is homeomorphic to a closed disc). The interior of  $F$  is the bounded component of  $\mathbb{C} - \mathcal{C}$  and  $F$  is equal to the closure of its interior.*

Observe that the dimension of  $\mathcal{C}$  is the same as the dimension of the von Koch curve.

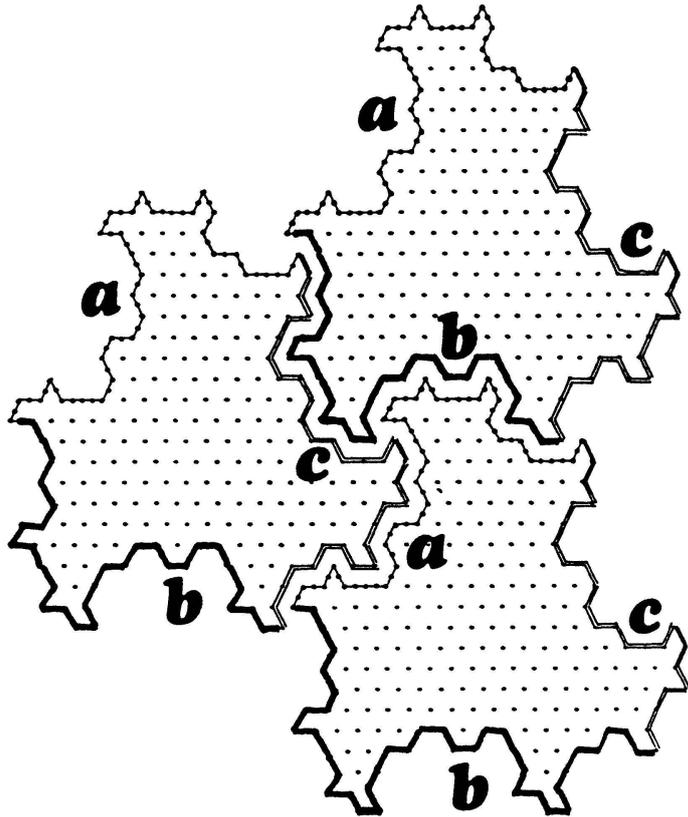


Fig. 5

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