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The Magic World of Geometry — I. The Isoperimetric Problem

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Nature is uncompromisingly effective, and from a geometrical point of view its shapes are full of mathematics. The study of optimal properties of geometric objects is therefore both interesting and important. In this article, I shall present the isoperimetric problem, which is an archetypical problem in this context. The article is the first in a series of three articles with the common subtitle "The Magic World of Geometry" intended to show how geometry appeals to the imagination and how it combines the concrete with the abstract.

Unter dem Sammeltitel *The Magic World of Geometry* hat V.L. Hansen drei Beiträge verfasst, deren erster hier vorliegt. Der Autor behandelt darin das isoperimetrische Problem in der Ebene: *Finde diejenige geschlossene Kurve in der Ebene, welche bei gegebener Länge den grössten Flächeninhalt einschliesst!* Er beschreibt den geometrischen Lösungsansatz des grossen Schweizer Mathematikers J. Steiner (1796–1863). Dieser Beweis hat den Vorteil grosser Anschaulichkeit, und er ist damit auch beispielhaft für Steiners Mathematik. Einige Jahre nach Steiners Veröffentlichung machte Weierstrass allerdings darauf aufmerksam, dass Steiner stillschweigend die Existenz einer Lösung vorausgesetzt hatte. Anschaulich scheint die Existenz einer Lösung zwar offensichtlich, aber Weierstrass konnte anhand ähnlicher Fragestellungen zeigen, dass sie nicht ohne weiteres angenommen werden darf. Die Beweislücke wurde später durch Weierstrass und Blaschke geschlossen. *ust*

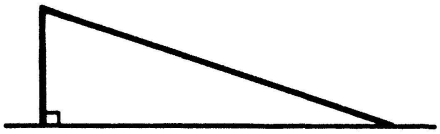

Fig. 1

Fig. 2

The isoperimetric problem is one of the classical geometric variational problems. In its simplest form the problem can be stated as follows: Find the closed plane curve (without self-intersections) of a fixed prescribed length that encloses the maximal plane area.

The ancient Greeks took it for granted that the solution to the problem is what they regarded as the most perfect of all closed curves, namely the circle. However, not until 19th century was a complete proof given. Jacob Steiner (1796–1863) suggested several ingenious proofs, and we shall present one below.

Before we embark on Steiner's proof, it will be appropriate to make some general remarks concerning variational problems. Before Weierstrass (1815–1897) it was accepted without proof that an extremal problem, of a physical or geometrical nature, always has a solution. It is one of the many merits of Weierstrass to have pointed out that this is by no means the case. In a primitive, numerical way we can see this already from the real numbers. For example, the set of fractions

$$\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$$

does not contain a smallest positive number.

Weierstrass gave several simple examples of geometrical problems without a "minimal" solution. In Figures 1 and 2 we give two such examples. In Figure 1 we consider two points A and B on a line, and seek the shortest polygonal path in the plane, which starts in A orthogonally to the line and ends in B. Obviously, there is no such shortest path. In Figure 2 we consider two points on a line through the origin in the plane. If we remove the origin from the plane, there is no longer a shortest path connecting the two points.

It is instructive to learn that during the history of mathematics, even great mathematicians like Dirichlet (1805–1859) and Riemann (1826–1866) have overlooked that the fundamental problem can be hidden where it is least expected. In the calculus of variations the main problem often is the bare question of the existence of an optimal object.

In famous lectures at the University of Berlin in the 1870's, Weierstrass developed general methods to ensure the existence of global maxima and minima of continuous functionals. Using these methods it is not difficult to prove that the isoperimetric problem has a solution if we restrict our attention to n-gons, i.e. closed plane polygonal curves without self-intersections and with n edges. First note that in a coordinate system in the plane, an n-gon can be described by the $2n$ coordinates of its corners. Observe next that all possible shapes of an n-gon with a fixed prescribed length can be described by the coordinates in a closed and bounded subset of $2n$-dimensional real number space. Since the area of an n-gon depends continuously on the $2n$ coordinates of its corners, the area attains a maximum value on this closed and bounded set according to a fundamental theorem

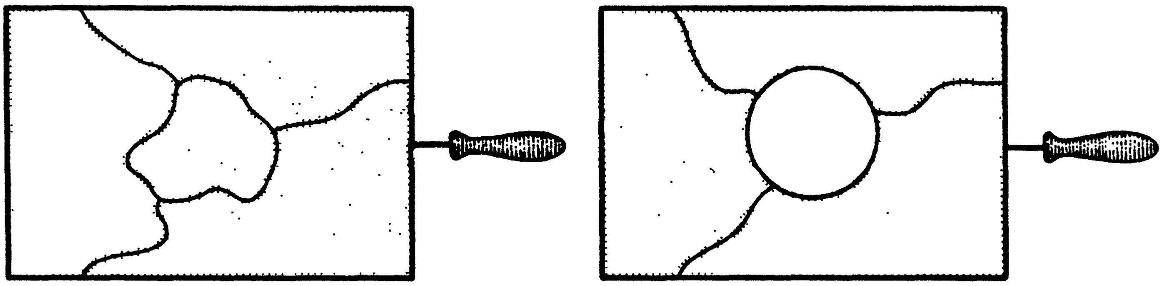

Fig. 3

of Weierstrass. This proves the existence of a solution to the restricted isoperimetric problem. The existence of a solution to the general isoperimetric problem can now be proved by approximating an arbitrary closed (rectifiable) curve with polygonal curves. More details on the existence of a solution to the isoperimetric problem can be found in references ([1], §5 and §8), ([2], Chp. IV, §2.1, p. 149) and ([3], Chp. VII, §8, p. 375).

It is easy to illustrate by the following experiment that the circle is the solution to the isoperimetric problem. Suspend a closed curve in a small frame. The curve and the threads with which it is attached to the frame can e.g. be sewing thread. Now dip the frame in a suitable soap solution. Thereby the closed curve will be placed on a soap film. Prick a hole in the soap film inside the closed curve. Immediately the closed curve turns into a circle; cf. Figure 3. It follows that the circle must be the solution to the isoperimetric problem, since the potential energy in the soap film outside the curve is proportional to the area, so that the soap film will try to minimize the area, which on the other hand corresponds to the area inside the closed curve being maximized.

The physical experiment is pretty convincing but not sufficient evidence from a mathematical point of view. We could be misled by our senses. Maybe it is only very close to being a circle. Therefore we consider now a closed plane curve $\mathscr{C}$ without self-intersections and of a fixed prescribed length L with the property that, among all such curves, it encloses the largest plane area. Following Steiner, we shall then argue mathematically that $\mathscr{C}$ must be a circle. It should be mentioned that Steiner also took it for granted that the isoperimetric problem has a solution.

By considering Figure 4 it is clear that $\mathscr{C}$ must be convex, since if this was not the case we could create a closed curve of the same length L as $\mathscr{C}$, but enclosing a larger area, by reflecting an inbuckling to an outbuckling.

Now choose two points A and B on $\mathscr{C}$ that divide the curve into two arcs of equal length, i.e. length $L/2$. The chord AB divides the enclosed figure into two pieces, which each must have the same area since, otherwise, we could construct a figure of larger area by replacing the smaller piece with the mirror image in AB of the larger piece; cf. Figure 5.

Consider now the "half" figure bounded by one of the arcs from A to B of length $L/2$ and the line segment AB. Let P be an arbitrary point on the arc, and consider the triangle APB. Imagine that the arc is made of steel with a hinge at P. Without changing the length of the arc we can then bend the triangle at the corner P. Thereby it is easily seen, that the "half" figure has maximal area precisely when the triangle has a right angle at

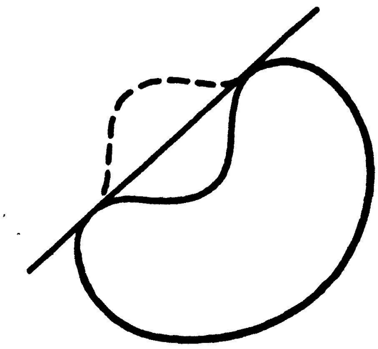

Fig. 4

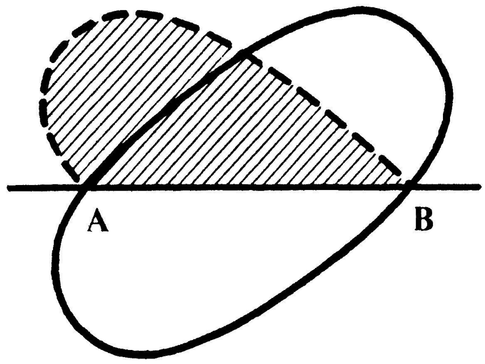

Fig. 5

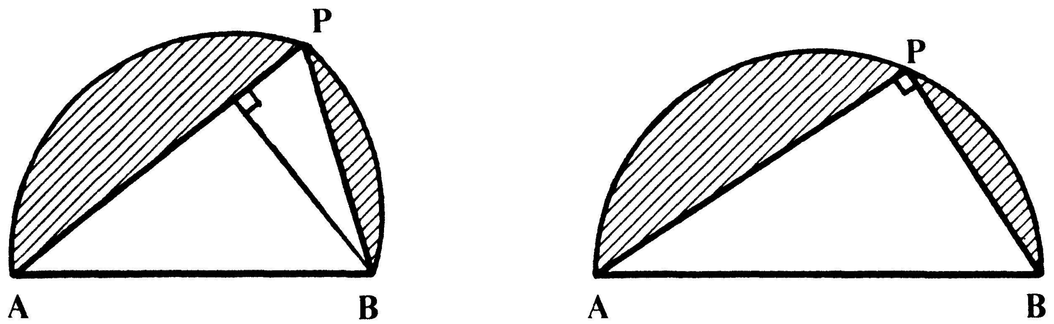

Fig. 6

P; cf. Figure 6. Since P was arbitrarily chosen, we conclude that the arc from A to B must be a semicircle, and the original closed curve $\mathscr{C}$ therefore a circle.

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