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## Densest Packing of Six Equal Circles in a Square

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Hans Melissen

Hans Melissen completed his studies at Utrecht University in 1982 with a masters degree thesis on partial differential equations. Since 1986 he is employed at Philips Electronics, where he is currently working at the Eindhoven Research Laboratories. His research interests include analysis and numerical treatment of Maxwell's equations, and computational geometry.

The problem of finding densest packings of  $n$  congruent circles inside a compact convex plane region has been investigated thoroughly during the past decades. In particular, a lot of work has been carried out on the determination of optimal circle packings in a circle, a square and an equilateral triangle for small values of  $n$  (cf. [2, 3, 16]).

In 1967, circle packings inside a circular disc were given for  $n = 2, \dots, 16$  by S. Kravitz ([9]). The optimality of these configurations for  $n \leq 7$  was proved by R.L. Graham ([1]), and U. Pirl ([19]) gave proofs for  $n \leq 10$ . Pirl also made some conjectures for  $11 \leq n \leq 19$ . Subsequent improvements were obtained for  $n = 14, 16, 17$  and  $20$  ([4]) and again for  $n = 17$  ([20]). The optimality for  $n = 11$  was proved recently by the author ([11]).

Densest packings of  $n$  circles in an equilateral triangle are known for the triangular numbers  $n = k(k+1)/2$  (see [17]) and for  $n \leq 12$  ([10, 13]). Further conjectures are given in [11, 12].

The problem of optimally packing circles into a square was raised for  $n = 8$  by L. Moser ([15]). The optimality of the conjectured packing was proved by J. Schaer and A. Meir ([23]). Schaer ([22]) also solved the problem for  $n = 9$  and gave configurations for  $n \leq 7$ . He remarked that the cases  $n = 2, 3, 4$  and  $5$  'are solved easily', and that  $n = 6$

Hans Melissen nimmt hier das Problem des vorhergehenden Beitrages noch einmal auf: Es sollen 6 gleiche Kreise mit möglichst grossem Radius in einem Quadrat plaziert werden. An mehreren Stellen in der Literatur wird die optimale Anordnung für 6 Kreise ohne Beweis und ohne Hinweis auf eine Quelle beschrieben. Nach der Drucklegung dieses Beitrages hat Hans Melissen allerdings festgestellt, dass B.L. Schwartz 1970 einen entsprechenden Beweis geliefert hat; siehe dazu die Notiz am Ende des Beitrages. *ust*

had been proved by R.L. Graham. A geometric outline of a proof for  $n = 7$  exists only in the form of an unpublished manuscript ([21]). Later, G. Wengerodt ([27, 28, 29]) proved the cases  $n = 14, 16, 25$ , and  $n = 36$  was solved by K. Kirchner and G. Wengerodt ([8]). Recently, computer assisted proofs for  $10 \leq n \leq 20$  have been described by Peikert et al. ([7, 18]). For other values of  $n \leq 27$  candidates for optimal packings have been given in ([5, 14, 24, 25]).

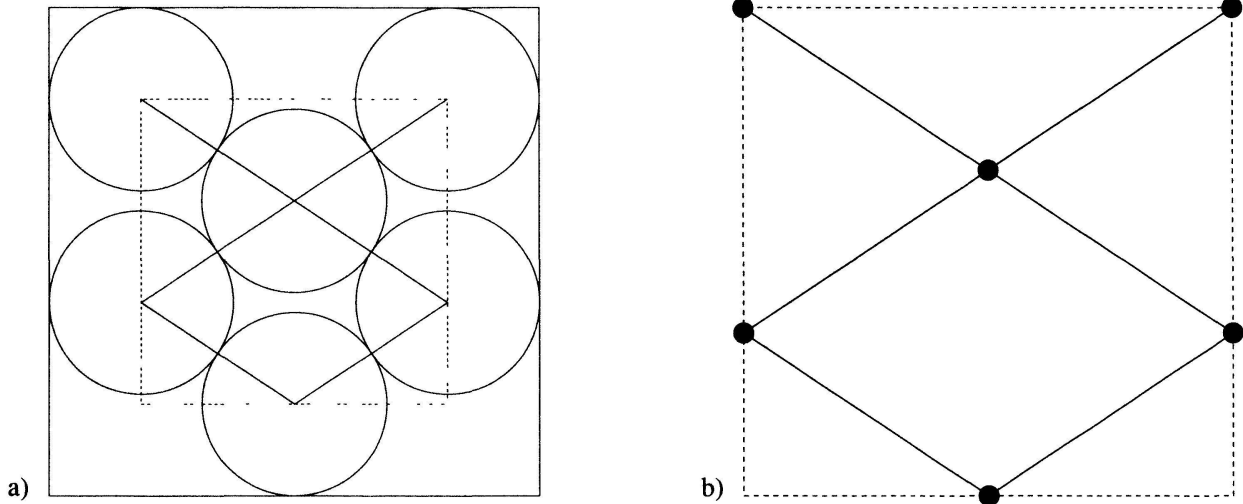


Fig. 1 a) Closest packing of six equal circles in a square.  
b) Maximum least distance arrangement for six points in a square. The solid line segments between the points are of equal length.

A useful, often employed fact is that finding a densest packing of  $n$  equal circles in a circle, a square or a triangle is equivalent to positioning  $n$  points inside that set such that the minimum distance between the points is maximal (see for instance Figure 1a). We shall use this last formulation.

The optimal configuration for six points in a square (up to rotations) is shown in Figure 1b. The minimum distance between the points is  $d_6 = \sqrt{13}/6$ . The proof of this case was attributed to Graham by Schaer ([22]). It was probably given in a private letter, but, unfortunately, it has never been published, and no further notes exist ([6]). In [2] the desirability of a proof for  $n = 6$  was also mentioned. In this paper we will provide such a proof.

The proof is based on the partition of the unit square  $[0, 1]^2$  into nine smaller regions as indicated in Figure 2. The partition is completely determined by the distances  $|p_8p_{10}| = |p_{10}p_{11}| = 1/3$ ,  $|p_5p_8| = d_6$  and the obvious symmetries in the diagonals. The diameter of each of the subregions does not exceed  $d_6$ . Suppose that we have a configuration  $\mathcal{N} = \{x_1, x_2, \dots, x_6\}$  of six points in the square for which the minimum distance between the points is equal to  $d \geq d_6$ . Then each subregion can contain at most one point of this configuration. This is a result of the particular way in which the boundaries are distributed over the subregions, as is indicated by the dashed/solid lines in Figure 2.

First, we note that if there is a point of  $\mathcal{N}$  in a  $B$ -region as well as points in both of its neighbouring  $A$ -regions ( $\mathcal{N}$  will then be said to have the ‘ $ABA$ -property’), then  $d$  is equal to  $d_6$ . This can be seen, for instance for  $A_1, B_1$  and  $A_2$ , by subdividing the union of these three regions into two regions of diameter  $d_6$  with a cut along  $p_1p_4$  and applying

Dirichlet's pigeon-hole principle ( $p_1$  bisects the lower edge of the square): two of the points must be in the same region, so  $d \leq d_6$ . It is clear that the points from  $\mathcal{N}$  can only be  $p_1, p_2$  and  $p_3$ .

We will now consider the two situations in which there is either a point of the configuration in the region  $C$ , or  $\mathcal{N} \cap C$  is empty.

1. First, suppose that  $\mathcal{N} \cap C$  is empty. If there are three or four  $A$ -regions containing points from  $\mathcal{N}$ , then  $\mathcal{N}$  has the  $ABA$ -property. The remaining alternative is that only two of the  $A_j$  each contain a point of the configuration. All four  $B$ -regions must then also contain a point. The only situation that is not  $ABA$  is where the two  $A$ -regions are opposite with respect to  $C$ . If, for instance, there is both a point in  $A_1$  and  $A_3$ , then the point in  $A_3$  is restricted to a small neighbourhood of  $p_9$ , due to the presence of points in  $B_2$  and  $B_3$ . This in turn restricts the position of the point in  $B_2$ . A similar restriction holds for the point in  $B_1$ . It is easy to verify that these two points then lie too close together, so this situation cannot occur.

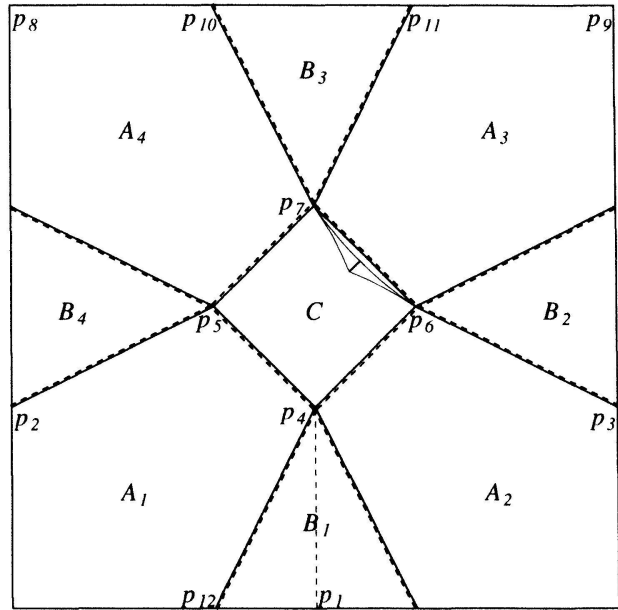


Fig. 2 Partition of the square. The dashed/solid lines indicate to which region each edge is assigned.

2. Secondly, suppose that there is a solution point in region  $C$ . It is not possible that two opposite  $B$ -regions, like for instance  $B_1$  and  $B_3$ , both contain a point of  $\mathcal{N}$ . This is seen by dividing the union of  $B_1$ ,  $C$  and  $B_3$  with a cut along  $p_5p_6$  into two regions of diameter  $d_6$ . It means that at most two  $B$ -regions can contain a point of  $\mathcal{N}$ , so there must be at least three  $A$ -regions which contain a point of the configuration. Therefore we either have an  $ABA$ -situation, or a situation of the form where there is a point of  $\mathcal{N}$  in each of  $A_2, A_3, A_4, B_1$  and  $B_4$ . The latter situation is impossible as we will now show.

The three points in the regions  $A_3, B_1, B_4$  restrict the point in  $C$  to the small region bounded by three circle segments of radius  $d$  around  $p_2, p_9, p_{12}$  (see Figure 2). By symmetry, it is sufficient to consider only those positions above the diagonal through  $p_9$ . Let  $(x_j, y_j)$  ( $j = 1, 2, 3, 4$ ) denote the coordinates of the points from  $\mathcal{N}$  in the regions  $C, B_4, B_1$  and  $A_2$  respectively. The mutual geometric restrictions on the position of these points and the point in  $A_3$  then result in the following inequalities

$$\frac{1}{3} + \sqrt{d^2 - x_1^2} \leq y_1 \leq 1 - \sqrt{d^2 - (1 - x_1)^2}, \quad (1)$$

$$\frac{1}{3} \leq y_2 \leq y_1 - \sqrt{d^2 - x_1^2}, \quad (2)$$

$$\sqrt{d^2 - y_2^2} \leq x_3 \leq \frac{1}{2}, \quad (3)$$

$$\sqrt{d^2 - (1 - x_3)^2} \leq y_4 \leq \frac{1}{3}, \quad (4)$$

$$\frac{1}{2} \leq x_1 \leq 1 - \sqrt{d^2 - \left(\frac{1 - y_4}{2}\right)^2} \quad (5)$$

The inequalities in (1), for instance, result from the fact that the distances of  $(x_1, y_1)$  to  $p_2$  and  $p_9$  should at least be  $d_9$ . Initially we have

$$\frac{1}{2} \leq x_1 \leq 1 - \frac{d}{\sqrt{2}} \leq 1 - \frac{d_6}{\sqrt{2}}.$$

Now if we write  $x_1 = 1/2 + \varepsilon$ , this last inequality leads to  $0 \leq \varepsilon \leq (6 - \sqrt{26})/12$ . Some elementary estimates show that inequalities (1) ... (4) imply that

$$y_1 \leq \frac{2}{3} - \frac{7}{6}\varepsilon, \quad y_2 \leq \frac{1}{3} + \varepsilon, \quad x_3 \geq \frac{1}{2} - \frac{5}{6}\varepsilon, \quad y_4 \geq \frac{1}{3} - \frac{5}{3}\varepsilon.$$

If  $d > d_6$ , then all the above inequalities are strict. From (5) we see that

$$\frac{1}{2} + \varepsilon = x_1 \leq 1 - \sqrt{\frac{1}{4} - \frac{5}{9}\varepsilon - \frac{25}{36}\varepsilon^2} < \frac{1}{2} + \varepsilon,$$

if  $\varepsilon > 0$ . This shows that the only possible situation occurs for  $d = d_6$  and  $\varepsilon = 0$ . In this case the coordinates of the point in  $B_4$  would be  $(0, 1/3)$ . By the choice of boundary, however, this point is not in  $B_4$ , which shows that this situation is impossible.

We have shown that  $d_6$  is optimal. From the proof it follows that we always end up with an ABA-situation. Suppose for instance that there are three points of  $\mathcal{N}$  in  $A_1$ ,  $B_1$  and  $A_2$ . These points can only be  $p_1, p_2, p_3$ . There can be no points in  $B_2$  or  $B_4$ , so there must be a point in  $C$ , because three points in  $A_3, B_3, A_4$  would not be compatible with  $p_2$  and  $p_3$ . The only feasible point in  $C$  is  $p_7$ , so the remaining two points must be  $p_8, p_9$ . This results in the solution depicted in Figure 1b.  $\square$

**Note:** After completion of the article it was found that an optimality proof for the packing of six circles in a square has been given previously by Schwartz ([26]). His proof uses similar techniques for a different partition.

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