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Another Computation of $\int_0^\infty e^{-u^2} du$

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Ronald Kortram was born in 1944. He studied mathematics in Leiden and obtained his PhD in 1971 under the direction of C. Visser. He worked at the universities of Helsinki and Nijmegen. His main interest is in the theory of functions of one complex variable.

There are many ways to determine the value of $\int_0^\infty e^{-u^2} du$. In this article we establish a relation between this value and the number of lattice points in a disc.

Let x be a real number, and let $0 < x < 1$. The function

$$t \to x^{t^2} = e^{t^2 \log x} \quad t \in [0, \infty)$$

is decreasing, thus

$$\int_0^\infty e^{t^2 \log x} dt \leq \sum_{n=0}^\infty x^{n^2} \leq 1 + \int_0^\infty e^{t^2 \log x} dt.$$

Substitution of $u = t\sqrt{-\log x}$ leads to

$$\int_0^\infty e^{-u^2} du \leq \sqrt{-\log x} \sum_{n=0}^\infty x^{n^2} \leq \sqrt{-\log x} + \int_0^\infty e^{-u^2} du,$$

thus

$$\lim_{x \uparrow 1} \sqrt{-\log x} \sum_{n=0}^\infty x^{n^2} = \int_0^\infty e^{-u^2} du.$$

Die Berechnung des Gaussschen Fehlerintegrals $\int_{-\infty}^{+\infty} e^{-u^2} du$ ist üblicherweise einer der Höhepunkte in der einführenden Vorlesung über Differential- und Integralrechnung, und zwar für alle Beteiligten und unabhängig von der benützten Methode. Selbst spontaner Applaus von Seiten der Zuhörerschaft ist an dieser Stelle keine Seltenheit! — In seinem Beitrag behandelt Ronald Kortram einen offenbar neuen Zugang zur Berechnung dieses Integrals, bei dem sich überraschende Beziehungen zum bekannten zahlentheoretischen Problem über die Darstellung von ganzen Zahlen als Summe von zwei Quadraten zeigen. *ust*

From $\lim\limits_{x\to 1} \frac{\log x}{x-1} = 1$, we conclude that

$$\int_0^\infty e^{-u^2} du = \lim_{x\uparrow 1} \sqrt{1-x} \sum_{n=0}^\infty x^{n^2},$$

i.e.

$$\left(\int_0^\infty e^{-u^2} du\right)^2 = \lim_{x\uparrow 1} (1-x) \left(\sum_{n=0}^\infty x^{n^2}\right)^2 = \lim_{x\uparrow 1} (1-x) \sum_{n=0}^\infty b_n x^n$$

where b_n is the number of representations of n as sum of two squares of non-negative integers, i.e. b_n is the number of lattice points with non-negative coördinates, at distance $\sqrt{n}$ from the origin.

We base the computation of $\lim\limits_{x\uparrow 1}(1-x) \sum\limits_{n=0}^\infty b_n x^n$ on the following lemma.

Lemma: *Let A_n be a sequence of positive numbers, such that $\sum A_n x^n$ converges for $x \in [0,1)$ but diverges for $x = 1$. Let B_n be a sequence of numbers such that*

$$\lim_{n\to\infty} \frac{B_n}{A_n} = \lambda.$$

Then we have

$$\lim_{x\uparrow 1} \frac{\sum\limits_{n=0}^\infty B_n x^n}{\sum\limits_{n=0}^\infty A_n x^n} = \lambda.$$

Proof: Since $|B_n|$ is dominated by a constant multiple of A_n, $\sum B_n x^n$ converges for $x \in [0,1)$. Let $\varepsilon > 0$ be given. Choose N such that for all $n > N$ we have

$$\left|\frac{B_n}{A_n} - \lambda\right| < \frac{1}{2}\varepsilon,$$

i.e.

$$|B_n - \lambda A_n| < \frac{1}{2}\varepsilon A_n.$$

For $x \in [0,1)$ it follows that

$$\left|\frac{\sum\limits_{n=0}^\infty B_n x^n}{\sum\limits_{n=0}^\infty A_n x^n} - \lambda\right| = \frac{\left|\sum\limits_{n=0}^\infty (B_n - \lambda A_n) x^n\right|}{\sum\limits_{n=0}^\infty A_n x^n} \le \frac{\sum\limits_{n=0}^N |B_n - \lambda A_n| + \frac{1}{2}\varepsilon \sum\limits_{n=N+1}^\infty A_n x^n}{\sum\limits_{n=0}^\infty A_n x^n}$$

$$< \frac{\sum\limits_{n=0}^N |B_n - \lambda A_n|}{\sum\limits_{n=0}^\infty A_n x^n} + \frac{1}{2}\varepsilon,$$

and for x sufficiently close to 1, this is smaller than ε, and the lemma is proved. □

We apply this lemma, with

$$A_n = n+1 \qquad \text{and} \qquad B_n = \sum_{k=0}^{n} b_k,$$

the number of lattice points with non-negative coördinates in the closed disc with radius $\sqrt{n}$ around the origin. To each such lattice point we associate the unit square that lies "north east" of it. These squares cover a closed quarter-disc with radius $\sqrt{n}$ and are covered by a closed quarter-disc with radius $\sqrt{n}+\sqrt{2}$, hence

$$\frac{1}{4}\pi(\sqrt{n})^2 \le B_n \le \frac{1}{4}\pi(\sqrt{n}+\sqrt{2})^2$$

thus

$$\lim_{n\to\infty} \frac{B_n}{n+1} = \frac{1}{4}\pi.$$

It follows from the lemma that

$$\frac{1}{4}\pi = \lim_{x\uparrow 1} \frac{\sum_{n=0}^{\infty} B_n x^n}{\sum_{n=0}^{\infty}(n+1)x^n} = \lim_{x\uparrow 1} \frac{(1-x)\sum_{n=0}^{\infty} B_n x^n}{(1-x)\sum_{n=0}^{\infty}(n+1)x^n} =$$

$$= \lim_{x\uparrow 1} \frac{\sum_{n=0}^{\infty} b_n x^n}{\sum_{n=0}^{\infty} x^n} = \lim_{x\uparrow 1}(1-x)\sum_{n=0}^{\infty} b_n x^n,$$

and this shows that

$$\int_0^{\infty} e^{-u^2}\,du = \frac{1}{2}\sqrt{\pi}.$$

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