# When is a polygon circumscribing a regular polygon again regular? 

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# When is a Polygon Circumscribing a Regular Polygon Again Regular? 


#### Abstract

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## Introduction

In the March 1970 issue of "Mathematics Magazine" the following problem appeared: Is the triangle $\triangle A B C$ in Fig. 1a equilateral? Four different solutions to this problem appeared in its November 1970 issue, and shortly thereafter the English geometer J.F. Rigby generalized the problem to other polygons [2], [3]. ${ }^{2}$ ) In particular, he showed that the

Gegeben sel ein $n$-Eck $P$ mit Eckpunkten $P_{1}, P_{2, \ldots} P_{n}$. Auf den Seiten $P_{i m 1} P_{i}$, $i=1,2, \ldots, n\left(P_{0}=P_{n}\right)$ seien Punkte $A_{i}$ festgelegt mit $A_{1} P_{1}=A_{2} P_{2}=\ldots=A_{n} P_{n}$ Ist P ein reguläres Polygon, so ist offensichtich das aus den Punkten $A_{1}, A_{2, \ldots, *} A_{n}$ gebildete Polygon A ebenfalls regulär. Der vorliegende Beitrag beschätigt sich mit der Umkehrung dieses Schlusses: Folgt aus der Regularităt des Polygons A auch die Regularität von P? Bekannte Resultate (fur $n=3,4$, and $n \geq 6$ gerade) werden hier ergänzt und präzisient. Dabei verdient der Weg, den die Autoren einschlagen, selbstandiges In teresse. Das geometrische Problem wird in cine analytische Fragestellung Ibersetz, die im Rahmen dynamischer Systeme interpretiert wirds. Es ist die Frage zu beantworten, ob eine gegebene reelle Funktion einen periodischen Punkt, dih. ob das zugehörige dynamische System eine periodische Bahn besitzt, In dieser interpretation lassen sich schliessich auch experimentell Informationen fir den Fall $n=5$ gewinnen, der in diesem Problem eine merkwurdige Sonderrolle zu spielen scheint, ust

[^0]quadrilateral circumscribing the square must itself be a square, but that there exists a non-regular circumscribing hexagon (Fig. 1b).


Fig. 1 The circumscribing hexagon has "period 2" (see Th. 1). The angles of $\triangle A_{i} P_{i} A_{i+1}$ are $30^{\circ}, 100^{\circ}$, $50^{\circ}$ for $i$ even and $10^{\circ}, 140^{\circ}, 30^{\circ}$ for $i$ odd.

To be precise, let $\mathbf{A}=A_{1} \ldots A_{n}$ and $\mathbf{P}=P_{1} \ldots P_{n}$ ( $A_{i}$ and $P_{i}$ labelled counterclockwise) be convex $n$-sided polygons (abbr. $n$-gons). Suppose that $\mathbf{P}$ is circumscribed about $\mathbf{A}$ (i.e., $A_{i+1}$ lies between $P_{i}$ and $P_{i+1}$ ) with $A_{i} P_{i}=A_{i+1} P_{i+1}$, Fig. 1 b (we shall refer to $\mathbf{P}$ as a circumscribing polygon). This paper is concerned with the question: if $\mathbf{A}$ is regular, is $\mathbf{P}$ necessarily regular?
Besides the cases already mentioned, Rigby proved that the answer is negative for $n \geq 6$ even. In this paper we answer the question for the remaining values of $n$ (Theorem 1 , part 1). In addition, we analyze the types of non-regular polygons that arise (Theorem 1, part 2).
The main results of this paper are given in the following theorem.
Theorem 1 Suppose a regular n-gon $\mathbf{A}$ with sides equal to one is inscribed in an $n$-gon $\mathbf{P}$ with $A_{1} P_{1}=A_{2} P_{2}=\cdots=A_{n} P_{n}$. Then

1. if $n=3,4$ or $n \geq 7$ odd, then $\mathbf{P}$ must be regular;
2. if $n \geq 6$ even, then $\mathbf{P}$ may be non-regular. Moreover, for $n \geq 8$, the angles and sides of $\mathbf{P}$ must have period 2, i.e.: $\angle P_{1}=\angle P_{3}=\cdots$ and $\angle P_{2}=\angle P_{4}=\cdots ; P_{1} P_{2}=P_{3} P_{4}=$ $P_{5} P_{6}=\cdots$ and $P_{2} P_{3}=P_{4} P_{5}=\cdots$

Our approach differs markedly from Rigby's: a new theme of abundant and deficient angles unifies our proof of part 1 of Theorem 1 . The basic idea here is quite simple. It is shown in Lemma 2 that for $n \geq 6$ the angles of a non-regular circumscribing $n$-gon $\mathbf{P}$ must alternate between being larger than (abundant) and smaller than (deficient) the angle of a regular $n$-gon, and thus can only occur if $n$ is even. A similar argument works for $n=3$ and 4 .

To prove the second part of Theorem 1 we translate the geometric question (whether there are non-regular $n$-gons of period $k$ ) to an analytic one, namely: Does a certain function $f$ have a periodic point with period equal to $k$ ( $k>1$ a divisor of $n$ )? In Theorem 2 the existence of period 2 points is proved, and in Theorem 3 it is shown that there are no points of period greater than 2 if $n>6$.
The paper is divided into two sections which contain the proofs of the first and second parts of Theorem 1, respectively. The dynamics of the polygons $\mathbf{P}$ are discussed in Remark 3, and a bifurcation diagram is given for the case of the hexagon (Fig. 6). A picture of a non-regular pentagon along with graphs are given in Remark 4 at the end of the paper.

## 1 The Cases in Which There are Only Regular Circumscribing Polygons

Let $\mathbf{A}$ and $\mathbf{P}$ be as in Theorem 1. Let $\alpha$ denote $\frac{(n-2) \pi}{n}$, the angle measure of a regular $n$-gon. Central to our discussion is the following notion for the angles of $\mathbf{P}$ : the angle $\angle P_{i}$ is abundant (resp. deficient) if $\angle P_{i}>\alpha$ (resp. $\angle P_{i}<\alpha$ ).
First we observe that if $\angle P_{i}=\alpha$, for $1 \leq i \leq n$, then $\mathbf{P}$ must be regular. (The triangle case is clear. If $n \geq 4$, noting that $\alpha$ is not acute we have $\triangle A_{1} P_{1} A_{2} \cong \triangle A_{2} P_{2} A_{3}$. Hence $P_{1} A_{2}=P_{2} A_{3}$ and $P_{1} P_{2}=P_{1} A_{2}+A_{2} P_{2}=P_{2} A_{3}+A_{3} P_{3}=P_{2} P_{3}$.) Thus a non-regular circumscribing polygon $\mathbf{P}$ must have both deficient and abundant angles since the sum of its angles is $n \alpha$.
Geometrically it is easy to determine whether $\angle P_{i}$ is abundant or deficient. Given two points $A$ and $B$, the locus of points $P$ with $\angle A P B$ equal to a given angle measure are two circular arcs, one on each side of the segment $\overline{A B}$. When $A, B$ are $A_{i}, A_{i+1}$, and the given angle measure is $\alpha$, the arc which lies outside (= the side opposite the center of A) of the segment $\overline{A_{i} A_{i+1}}$ will be referred to as the regular arc on $\overline{A_{i} A_{i+1}}$ and denoted by $\widehat{A_{i} A_{i+1}}$. (See Figs. 2, 3, and 4.) If $P_{i}$ lies inside (resp. outside) the regular arc $\widehat{A_{i}} \widehat{A_{i+1}}$ then $\angle P_{i}>\alpha$ (resp. $\angle P_{i}<\alpha$ ). Thus $\angle P_{i}$ is abundant (resp. deficient) if and only if $P_{i}$ lies inside (resp. outside) the regular arc $\widehat{A_{i} A_{i+1}}$.

Next we show that a regular arc and its
 neighboring sides are tangent. Suppose $n \geq$ 5, then the rays $\overrightarrow{A_{i-1} A_{i}}$ and $\overrightarrow{A_{i+2} A_{i+1}}$ intersect, say at $Q$. Suppose $\overrightarrow{O Q}$ intersects the regular arc $\widehat{A_{i} A_{i+1}}$ at $R_{i}$, where $O$ is the center of the circle containing the regular $\operatorname{arc} \widehat{A_{i} A_{i+1}}$ (Fig. 2).

Fig. $2 \quad(n=6)$

Then

$$
\begin{align*}
& \angle A_{i} Q A_{i+1}=\pi-2 \angle Q A_{i} A_{i+1}=\pi-2(\pi-\alpha)=2 \alpha-\pi=\frac{n-4}{n} \pi .  \tag{1}\\
& \angle A_{i} O R_{i}=\pi-2 \angle A_{i} R_{i} O=\pi-\angle A_{i} R_{i} A_{i+1}=\pi-\alpha .
\end{align*}
$$

Using these two equations we get

$$
\angle O A_{i} Q=\pi-\frac{1}{2} \angle A_{i} Q A_{i+1}-\angle A_{i} O R_{i}=\pi-\left(\alpha-\frac{\pi}{2}\right)-(\pi-\alpha)=\frac{\pi}{2}
$$

This proves the following lemma for $n \geq 5$ :
Lemma 1 Each regular arc $\widehat{A_{i} A_{i+1}}$ is tangent to its two neighboring sides $\widehat{A_{i-1} A_{i}}$ and $A_{i+1} A_{i+2}$ at $A_{i}$ and $A_{i+1}$, respectively.

In fact it is easy to check that this Lemma also holds for $n=3$ and 4 .
We make a simple but crucial observation (needed in the proof of the following key Lemma 2). Suppose $R_{i}$ lies on the regular arc $\widehat{A_{i} A_{i+1}}$. Join $R_{i}$ to $A_{i+1}$ and extend it until it meets the next regular arc $\widehat{A}_{i+1} \widehat{A}_{i+2}$, say at $R_{i+1}$ (Fig. 3a). Then $A_{i} R_{i}=A_{i+1} R_{i+1}$. (Proof. Since $\angle A_{i+1}=\alpha=\angle R_{i}$ implies $\angle A_{i+1} A_{i} R_{i}=\angle A_{i+2} A_{i+1} R_{i+1}$, we get by the side-angle-angle theorem that $\triangle A_{i} R_{i} A_{i+1} \cong \triangle A_{i+1} R_{i+1} A_{i+2}$.)

Fig. 3 a)


Fig. 3 b)


Lemma 2 Let $\mathbf{P}$ be a circumscribing $n$-gon with $n \geq 6$. If $\angle P_{i}$ is deficient (resp.abundant) then $\angle P_{i+1}$ is abundant (resp. deficient).

Proof. Since $\mathbf{P}$ is convex, the vertex $P_{i}$ lies inside $\triangle A_{i} A_{i+1} Q$ (Fig. 2). In view of Eq. (1), $\angle P_{i}>\frac{n-4}{n} \pi$. On the other hand, if $\angle P_{i}$ is deficient then $P_{i}$ lies outside the regular arc $\widehat{A_{i} A_{i+1}}$ (Fig. 3a). Thus the line segment $\overline{P_{i} A_{i+1}}$ intersects $\widehat{A_{i} A_{i+1}}$, say at $R_{i}$. We have $\angle A_{i} R_{i} P_{i}=\pi-\alpha=\frac{2 \pi}{n}$ which is $\leq \frac{(n-4) \pi}{n}$ if $n \geq 6$. Thus $\angle P_{i}>\angle A_{i} R_{i} P_{i}$ and $A_{i} P_{i}<A_{i} R_{i}$. Construct $R_{i+1}$ from $R_{i}$ as in the above observation. Since $A_{i} R_{i}=A_{i+1} R_{i+1}$, the last inequality becomes $A_{i+1} P_{i+1}<A_{i+1} R_{i+1}$. Since $A_{i+1}, R_{i+1}$ and $P_{i+1}$ are collinear (they all lie on the line $\left.\overleftrightarrow{P_{i} A_{i+1}}\right), P_{i+1}$ lies between $A_{i+1}$ and $R_{i+1}$. So $P_{i+1}$ lies inside ${\widehat{A_{i+1}}}_{i+2}$ and $\angle P_{i+1}$ is abundant.
The case that $\angle P_{i}$ is abundant can be argued similarly (see Fig. 3b): We now have $P_{i}$ is between $R_{i}$ and $A_{i+1}\left(R_{i}, R_{i+1}\right.$ again lie on regular arcs) and $\angle R_{i}=\alpha>\frac{\pi}{2}>\angle A_{i} P_{i} R_{i}$, which implies $A_{i} R_{i}<A_{i} P_{i}$ or $A_{i+1} R_{i+1}<A_{i+1} P_{i+1}$. Thus $R_{i+1}$ is between $A_{i+1}$ and $P_{i+1}$ and $\angle P_{i+1}$ is deficient.
Remark 1. A similar argument yields: If $\angle P_{i}$ is deficient then $\angle P_{i+1}$ is deficient if $n=3$ or 4 , and can be either abundant or deficient if $n=5$. On the other hand, if $\angle P_{i}$ is abundant then $\angle P_{i+1}$ is abundant if $n=3$ and deficient if $n=4$ or 5 .
Proof of Theorem 1, part 1: if $n=3,4$ or $n \geq 7$ odd, then any circumscribing $n$-gon $\mathbf{P}$ must be regular.
Suppose $\mathbf{P}$ is a non-regular circumscribing polygon. By the observation at the beginning of this section, at least one $\angle P_{i}$ is deficient. If furthermore $n=3$ or 4 , by remark 1 , the existence of one deficient angle would imply that all angles are deficient, a contradiction. If on the other hand $n \geq 6$ then, by Lemma 2, the existence of one deficient angle implies that the angles alternate between abundant and deficient, which can happen only if $n$ is even.


We conclude this section with a remark about the range of the length $\ell:=A_{i} P_{i}$.
Remark 2. For a circumscribing polygon $\mathbf{P}$, regular or not, at least one $\angle P_{i}$ is $\geq \alpha$, which is $\geq \frac{\pi}{2}$ if $n \geq 4$. By considering $\triangle A_{i} P_{i} A_{i+1}$, we see that $0<\ell=A_{i} P_{i}<A_{i} A_{i+1}=1$. For $n=3$ the largest possible value of $\ell$ is the diameter $A_{i} R=\frac{2}{\sqrt{3}}$ of the circle containing the regular arc, see Fig. 4.
Note that for each $\ell$ in the above range a regular circumscribing $n$-gon $\mathbf{P}$ can be obtained by starting with $R_{1}$ on the regular arc with $A_{1} R_{1}=\ell$ and then construct $R_{2}, \ldots, R_{n}$ as in the observation before Lemma 2.
Thus there is a one-to-one correspondence between points on a regular arc and regular polygons $\mathbf{P}$.

## 2 The Cases of Non-Regular Circumscribing Polygons

In this section we will prove part 2 of Theorem 1. That is, it will be shown analytically that whenever $n \geq 6$ is even there are non-regular $n$-gons $\mathbf{P}$ satisfying the conditions of Theorem 1, and that, with the exception of the hexagon, all these polygons have period 2. In [2] and [3] a geometric proof is outlined for the existence of non-regular polygons, but our analytic approach leads to a description of the types of non-regular polygons $\mathbf{P}$. We adopt the same notation as in section 1 . Henceforth, $n \geq 6$, so by Remark 2 the length $\ell:=A_{i} P_{i}$ takes values in $(0,1)$.
We consider the function $f$ which for given $\ell$ computes $\theta_{i+1}$ from the argument $\theta_{i}$, where $\theta_{i}=\angle A_{i+1} A_{i} P_{i}$. A periodic orbit of $f$ of period $k$ corresponds to a polygon $\mathbf{P}$ of period $k$. More precisely, if $\phi$ is an angle between 0 and $2 \pi / n$ such that the $i$ th iterates $f^{i}(\phi)>0$ are distinct for $i=0, \ldots, k-1, f^{k}(\phi)=\phi$, and $n$ is divisible by $k$, then the sides and angles of the polygon $\mathbf{P}$ with $\angle A_{i+1} A_{i} P_{i}=f^{i}(\phi), i=1, \ldots, n$, have period $k$. (This is a consequence of basic theorems on triangles such as the side-angle-side theorem.)
Let $\psi_{i}=\angle A_{i} A_{i+1} P_{i}$. Then $\theta_{i+1}=\pi-\alpha-\psi_{i}=2 \pi / n-\psi_{i}$, where

$$
\psi_{i}=\arctan \left(\frac{P_{i} T}{A_{i} A_{i+1}-A_{i} T}\right)=\arctan \left(\frac{\ell \sin \theta_{i}}{1-\ell \cos \theta_{i}}\right)
$$

and $T$ is the foot of the perpendicular of $P_{i}$ onto the line $\overleftrightarrow{A_{i} A_{i+1}}$. Therefore, we have

$$
f(\theta)=\frac{2 \pi}{n}-\arctan \left(\frac{\ell \sin \theta}{1-\ell \cos \theta}\right)
$$

By Remark 2 there is exactly one regular polygon $\mathbf{P}$ for each $\ell$. For $\ell$ fixed, let $\beta$ represent the angle $\angle A_{2} A_{1} P_{1}$ of this regular polygon. Note that $f(\beta)=\beta$.

Lemma $3 f^{\prime}(\beta)<-1$ if and only if $\ell>\ell_{0}$, and $f^{\prime}(\beta)=-1$ only if $\ell=\ell_{0}$, where

$$
\begin{equation*}
\ell_{0}:=\left(4-3 \sin ^{2} \frac{2 \pi}{n}\right)^{-\frac{1}{2}} \tag{2}
\end{equation*}
$$

Proof. Let $S$ be the foot of the perpendicular of $A_{2}$ onto the line $\overleftarrow{A_{1} P_{1}}$ in the case in which $\mathbf{P}$ is regular. Since $n \geq 6, P_{1}$ lies between $A_{1}$ and $S$. Then

$$
\begin{equation*}
\ell=A_{1} S-P_{1} S=\cos \beta-c \sin \beta \tag{3}
\end{equation*}
$$

where $c=\cot \left(\angle A_{2} P_{1} S\right)=\cot (\pi-\alpha)=\cot (2 \pi / n)$.
The derivative of $f$ is given by

$$
\begin{equation*}
f^{\prime}(\theta)=\frac{\ell^{2}-\ell \cos \theta}{1-2 \ell \cos \theta+\ell^{2}} \tag{4}
\end{equation*}
$$

Since $1-2 \ell \cos \theta+\ell^{2}>0$, we have $2 \ell^{2}-3 \ell \cos \beta+1<0$ whenever $f^{\prime}(\beta)<-1$. Substituting Eq. (3) into the above inequality leads to $\cot \beta>2 c+1 / c$. Since the cotangent function is decreasing in the first quadrant we get $\beta<\operatorname{arccot}(2 c+1 / c)=: \gamma$. But $\cos \theta-c \sin \theta$ is decreasing in the first quadrant, so $\ell>\cos \gamma-c \sin \gamma=\left(c^{2}+1\right) / \sqrt{4 c^{4}+5 c^{2}+1}=$ $\sqrt{1+c^{2}} / \sqrt{1+4 c^{2}}$ which simplifies to $\ell_{0}$. Since the argument is valid if all the inequality signs are reversed (or replaced by equal signs) the assertion is proved.

Lemma 4 Let $h(\theta)=f^{2}(\theta)-\theta$, where $f^{2}$ is the composition off with itself. Then $h^{\prime}(\beta)>0$ if $\ell>\ell_{0}$, and $h^{\prime}(\beta)<0$ if $\ell<\ell_{0}$.

Proof. Since $h^{\prime}(\theta)=f^{\prime}(f(\theta)) f^{\prime}(\theta)-1$, we get $h^{\prime}(\beta)=\left[f^{\prime}(\beta)\right]^{2}-1$. Therefore, by the previous lemma, it suffices to show that $f^{\prime}(\beta)<1$. But this is equivalent to $\ell \cos \beta<1$. We are now ready to establish the second part of Theorem 1: the next theorem deals with period 2 circumscribing polygons, whereas Theorem 3 deals with higher periods.

Theorem 2 Let $n \geq 6$ be even. Then there exists a non-regular circumscribing polygon of period 2 for each $\ell, \ell_{0}<\ell<\ell_{1}$, where $\ell_{1}:=(2 \cos (2 \pi / n))^{-1}$, and $\ell_{0}$ is defined in Eq. (2). ${ }^{*}$ )

Based on plots of $f^{2}$, we conjecture that for each $\ell \in\left(\ell_{0}, \ell_{1}\right)$, there is only one period 2 circumscribing polygon, and for $\ell \notin\left(\ell_{0}, \ell_{1}\right)$, there are no period 2 circumscribing polygons (see Fig. 5). The complexity of $f^{2}$ prevents us from obtaining an analytic proof of this conjecture.


Fig. $5\left(n=8, \ell_{0} \approx 0.632, \ell_{1} \approx 0.707\right)$ Graphs of $f^{2}$ for $\ell=.37, .47, .57, .67, .77, .87$. The steeper graphs correspond to larger values of $\ell$.

Proof of Theorem 2. Since $h(\theta)=f^{2}(\theta)-\theta$, a zero $\phi>0$ of $h$ other than $\beta$ corresponds to a non-regular polygon $\mathbf{P}$ of period 2 as long as $f(\phi)>0$. By the definition of $\beta$,

[^1]$h(\beta)=0$. Moreover, by Lemma $4, h^{\prime}(\beta)>0$. Therefore, if $h(0)>0$ then $h$ will have at least one zero in the interval $(0, \beta)$ (by the intermediate value theorem). However, $h(0)=f(p), p=2 \pi / n$, so $h(0)>0$ is equivalent to $\tan p>\ell \sin p /(1-\ell \cos p)$. But this inequality holds if (and only if) $\ell<\ell_{1}$. (Note also that $f(\phi)>0$ since $\angle P_{1}$ is abundant, where $\phi=\angle A_{2} A_{1} P_{1}$.)

Remark 3. By examining plots of the function $f^{2}$, we are led to conjecture that as $\ell$ passes through $\ell_{0}$, the iteration goes through a period-doubling bifurcation (see [1], pp.158-159 for a discussion of period-doubling bifurcation). This is illustrated in Fig. 6.


Fig. $6(n=8)$ The dotted regular arc represents the vertices of regular polygons for $0<\ell<1$; the other curve consists of the two positions (for various $\ell$ ) of vertices of period-two polygons, one on each side of the regular arc. The bifurcation occurs at the intersection of the two curves.

Fig. 7a gives a nonregular hexagon (corresponding to an $\ell$ near $\ell_{0}$ ) and the regular hexagon (corresponding to $\ell_{0}$ ) from which the nonregular hexagon bifurcates. On the other hand, as $\ell$ increases to $\ell_{1}=1$, the non-regular hexagon approaches an equilateral triangle. This is illustrated in Fig. 7b. In general, the non-regular $2 m$-gon approaches a regular $m$-gon as $\ell$ increases to $\ell_{1}$.


Fig. 7a)


Fig. 7b)

Theorem 3 There are no non-regular $n$-gons, $n>6$, with period greater than two.

Proof. This is equivalent to there being no orbits of $f$ of period greater than 2 which are contained in the interval $(0, p), p=2 \pi / n, n>6$. Since a decreasing function cannot have a point with period greater than 2 , it suffices to show that for $0<\ell<1 f$ is decreasing on the subset of $(0, p)$ where $f>0$. In view of Eq. (4) $f^{\prime}(\theta) \leq 0$ if $\ell \leq \cos \theta$. By the first part of Theorem 1 we have $n \geq 8$, so $p \leq \pi / 4$. If $\ell \leq 1 / \sqrt{2}$ then $f^{\prime}(\theta) \leq 0$ on ( $0, p$ ).
Consider the case $1 / \sqrt{2}<\ell<1$. By the proof of Theorem 2, $f(p)<0$ if $\ell_{1}<\ell<1$. But $\ell_{1}=(2 \cos p)^{-1} \leq(2 \cos \pi / 4)^{-1}=1 / \sqrt{2}$ for $n \geq 8$, so $f(p)<0$ in this case. However, $f$ is concave up on ( $0, p$ ) since

$$
f^{\prime \prime}(\theta)=\frac{\sin \theta\left(\ell-\ell^{3}\right)}{\left(1-2 \ell \cos \theta+\ell^{2}\right)^{2}}
$$

is positive for $0<\theta<\pi$. Thus, in this case, the concavity of $f$ and the fact that $f(p)<0$ imply that $f$ is decreasing on the subinterval of $(0, p)$ on which $f>0$.

Remark 4. The results so far still leave unanswered the question whether there are nonregular pentagons and non-regular hexagons of period 6. (Non-regular hexagons of period 3 are precluded since, by Lemma 2, deficient and abundant angles must alternate.) By examining plots of the function $f$ and its iterates we have concluded that there are nonregular pentagons (Fig. 8) but no non-regular hexagons of period 6 (Fig. 9). Indeed, there seems to be a range of $\ell$ for each value of which there are $t w o$ non-regular pentagons.

a)

b)

Fig. $8(n=5)$ Graphs of $f$ and $f^{5}$ for $\ell=.946$. Note that $f^{5}$ has eleven fixed points so there are two non-regular pentagons for this value of $\ell$.


Fig. $9 \quad(n=6)$ Graphs of $f$ and $f^{6}$ for (a) $\ell=.77$ and (b) $\ell=86$.

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[^0]:    1) This work is partially supported by California State University, Long Beach.
    2) Rigby also treated other types of "circumscribed polygons" and the analogous question for the hyperbolic plane.
[^1]:    *) $\ell_{1}$ can be interpreted geometrically as $A_{i} Q$, where $Q$ is defined in Fig. 2. Also a simple check verifies that $\ell_{0}<\ell_{1}$. Moreover, both $\left\{\ell_{0}(n)\right\}$ and $\left\{\ell_{1}(n)\right\}$ are decreasing sequences which converge to one-half, and $\ell_{1}-\ell_{0}=O\left(n^{-2}\right)$.

