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Geometrical Aspects of the Circular Billiard Problem

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Norbert Hungerbühler wurde 1964 geboren. Er studierte Mathematik an der ETH Zürich, wo er 1989 diplomierte. Zur Zeit arbeitet er an einer Dissertation auf dem Gebiet der nichtlinearen partiellen Differentialgleichungen.

Recently Jörg Waldvogel investigated the problem of the circular billiard (see [Wa]). Let us recall the setting of the problem: Given two points $a, b \in \mathbb{C}$, $|a| < 1$, $|b| < 1$ in the interior of the unit disc, find all reflection points $\zeta = e^{i\varphi}$ on the unit circle such that the segments $\zeta - a$ and $\zeta - b$ describe the path of a bouncing billiard ball. We assume that a, b and the center o of the unit circle S are not collinear and $|a| \neq |b|$ since these cases are trivial.

We define L as the locus of all points $p \in L \subset \mathbb{C}$ such that the angles $\sphericalangle(a,p,o)$ and $\sphericalangle(o,p,b)$ are equal (see Figure 1). Hence the reflection points ζ are the points of intersection of L with the unit circle S. In the sequel let $\tilde{}$ denote the inversion with respect to the unit circle: $z \mapsto \tilde{z} := 1/\bar{z}$. Since S is invariant under the inversion mapping, the reflection points ζ are equal to $\tilde{L} \cap S$.

Using algebraic methods Waldvogel [Wa] proved the surprising fact that $\tilde{L}$ is an equilateral hyperbola. Here we obtain the same result by purely geometric arguments.

Lemma $\tilde{L}$ is the locus of all points $\tilde{p}$ of the plane such that

$$\sphericalangle(\tilde{p},\tilde{a},o) = \sphericalangle(o,\tilde{b},\tilde{p}).$$

Das Problem der idealisierten Billardkugel in einem Kreis wurde bereits im vorstehenden Beitrag von Jörg Waldvogel behandelt und zwar mit Hilfe von komplexen Zahlen und von einigen zusätzlichen geometrischen Überlegungen. Norbert Hungerbühler, angeregt durch die Arbeit von Jörg Waldvogel, geht hier dieselbe Frage mit rein geometrischen Mitteln an; es zeigt sich, dass die Benützung von Methoden der projektiven Geometrie, darunter auch des Satzes von Pascal, eine ausserordentlich elegante Behandlung des Problems ermöglichen. — Nach der Lektüre dieser beiden Beiträge drängt sich die Frage auf: Ist denn Billard wirklich nur ein Spiel? *ust*

Proof:

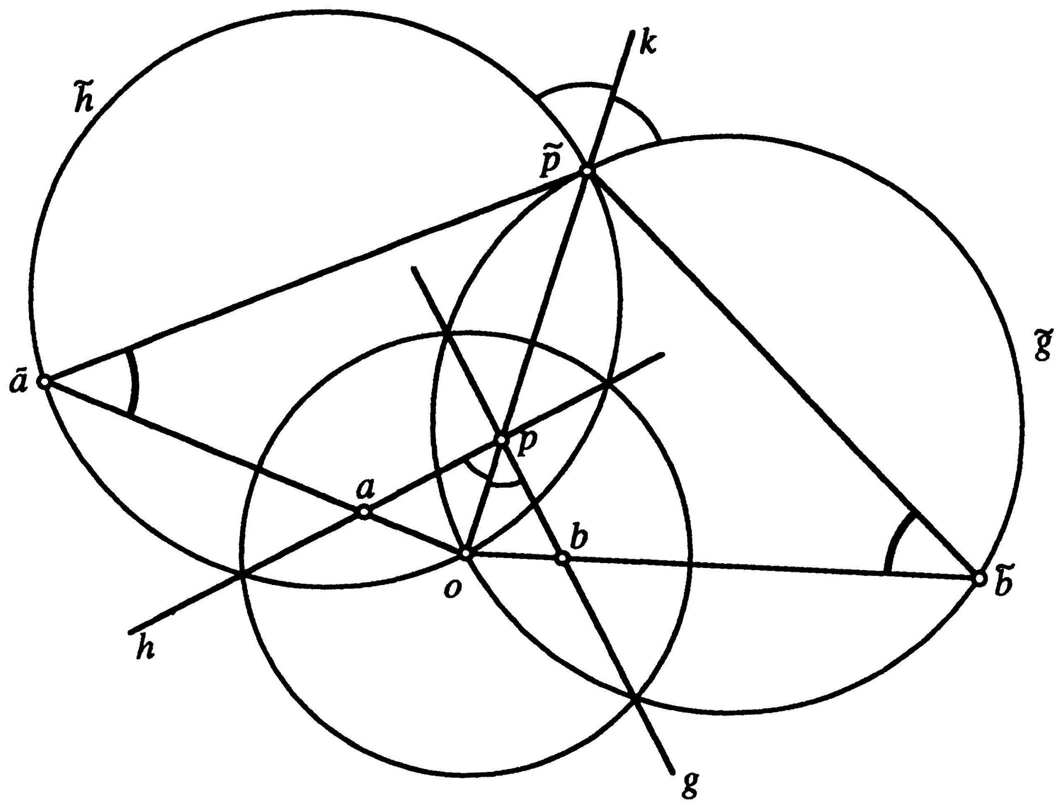

Fig. 1

Let $h := \overline{a\,p}$ be the line through the points a and $p \neq o$, $g := \overline{b\,p}$ and $k := \overline{o\,p}$. Hence $k = \tilde{k}$ and $\tilde{h}$ and $\tilde{g}$ are circles through $\tilde{a}$, o, $\tilde{p}$ and $\tilde{b}$, o, $\tilde{p}$ respectively. From $\sphericalangle(a,p,o) = \sphericalangle(o,p,b)$ follows $\sphericalangle(\tilde{h},\tilde{k}) = \sphericalangle(\tilde{k},\tilde{g})$ (the inversion preserves angles between curves: see Figure 1). Hence $\tilde{g}$ and $\tilde{h}$ have equal radii. This implies $\sphericalangle(\tilde{p},\tilde{a},o) = \sphericalangle(o,\tilde{b},\tilde{p})$. An analogous argument in the opposite direction finally proves the assertion.

Theorem *$\tilde{L}$ is an equilateral hyperbola through the points $\tilde{a}$, $\tilde{b}$ and o with asymptotes parallel to the (orthogonal) interior and exterior bisectors of the angle $\sphericalangle(\tilde{a},o,\tilde{b})$. The center of the hyperbola is the midpoint of the segment $\tilde{a}\,\tilde{b}$.*

Proof Let $\tilde{p}$ be a point of $\tilde{L}$. From the above lemma we know that $\sphericalangle(\tilde{p},\tilde{a},o) = \sphericalangle(o,\tilde{b},\tilde{p})$. Let w_1 be the interior and w_2 the exterior angle bisector of $\sphericalangle(\tilde{a},o,\tilde{b})$ and u_1 and u_2 be the ideal points of the projective plane in the direction of w_1 and w_2 respectively (see Figure 2). Let us reflect $\tilde{b}$ with respect to w_2 and denote the reflected point by q. Hence $q \in \overline{\tilde{a}\,o}$. Note, that $r := w_2 \cap \overline{\tilde{p}\,\tilde{b}}$ is not a point of the ideal line. Since $\sphericalangle(o,\tilde{b},r) = \sphericalangle(r,q,o)$ it follows that $\sphericalangle(o,\tilde{a},\tilde{p})$ and $\sphericalangle(r,q,o)$ are supplementary angles and hence that $\overline{\tilde{p}\,\tilde{a}}$ and $\overline{q\,r}$ are parallel. We conclude that in the hexagon $\tilde{p}\,\tilde{a}\,o\,u_2\,u_1\,\tilde{b}$ the intersections of the opposite sides $\overline{\tilde{a}\,o} \cap \overline{u_1\,\tilde{b}} = q$ and $\overline{o\,u_2} \cap \overline{\tilde{b}\,\tilde{p}} = r$ and $\overline{u_2\,u_1} \cap \overline{\tilde{p}\,\tilde{a}}$ are points of the projective line $\overline{q\,r}$. By the theorem of Pappus and Pascal it follows that $\tilde{p}$ is a point of the conic section through $\tilde{a}$, o, u_2, u_1, $\tilde{b}$. An analogous argument

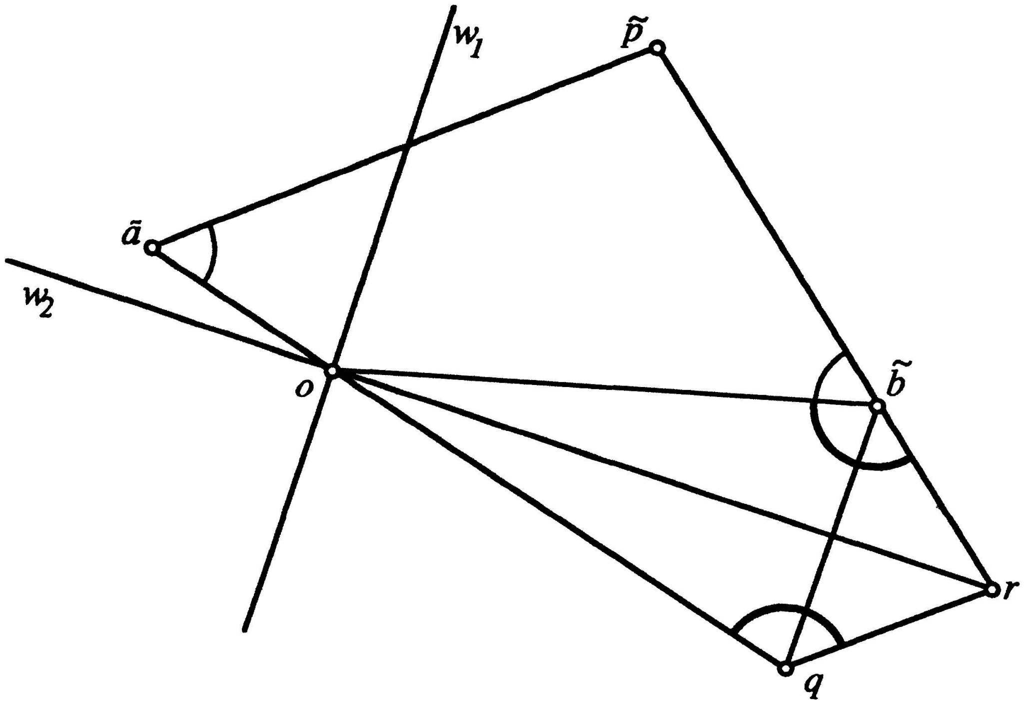

Fig. 2

in the opposite direction finally proves the first part of the assertion. In order to prove that the midpoint c of the segment $\tilde{a}\,\tilde{b}$ coincides with the center of the hyperbola, we show that $\tilde{L}$ is invariant under reflection with respect to c: Let $\tilde{p}$ be a point of $\tilde{L}$, hence $\sphericalangle(o,\tilde{a},\tilde{p}) = \sphericalangle(\tilde{p},\tilde{b},o)$. We reflect $\tilde{p}$ with respect to c and denote the reflected point with q. Since $\tilde{a}\,\tilde{p}\,\tilde{b}\,q$ is a parallelogram we get $\sphericalangle(q,\tilde{a},o) = \sphericalangle(o,\tilde{b},q)$ and hence $q \in \tilde{L}$ (see Figure 3).

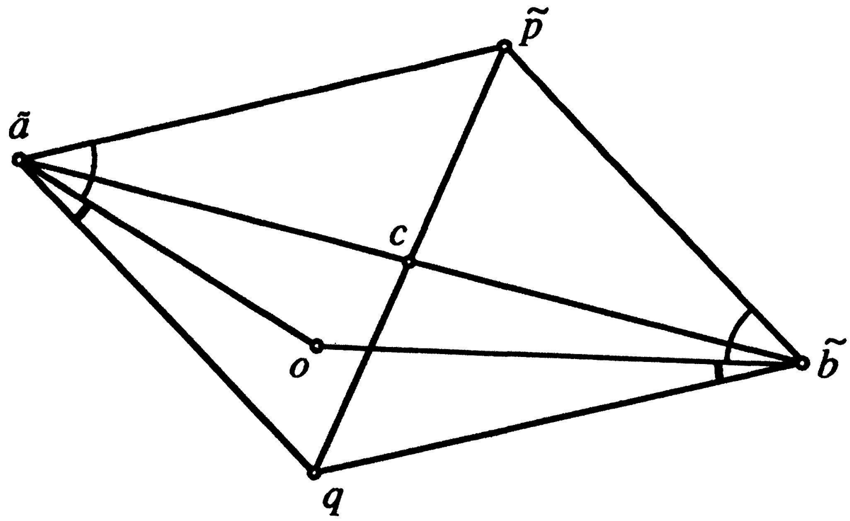

Fig. 3

Application Given a point a in the disc with radius r, $a \neq o$ the center of the disc. Find the path of a ball starting in a and after bouncing twice at the boundary S of the disc comming back to a.

Solution Note, that the arguments in the proofs of the lemma and the theorem hold without the condition $|a| < 1$, $|b| < 1$. Hence we solve the above problem by taking the point b as the ideal point in the direction orthogonal to $\overline{a\,o}$. Then $\tilde{L}$ is the equilateral hyperbola symmetric to $\overline{a\,o}$ through the points $\tilde{a}$ and o with center c being the midpoint of the segment $\tilde{a}\,o$. We denote $\{x,y\} := \tilde{L} \cap S$ and $p := \overline{a\,o} \cap \overline{x\,y}$, see Figure 4. Of course we have $\overline{a\,o} \perp \overline{x\,y}$ by symmetry. We get $|c\,p|^2 - |c\,o|^2 = r^2 - |o\,p|^2$ and hence $|c\,p|\,|o\,p| = (\frac{r}{\sqrt{2}})^2$. This leads to the following simple construction: Let d be the midpoint of $c\,o$ and q a point of the perpendicular to $\overline{c\,o}$ in c with $|c\,q| = \frac{r}{\sqrt{2}}$. Then $|d\,q| = |d\,p|$: see Figure 4.

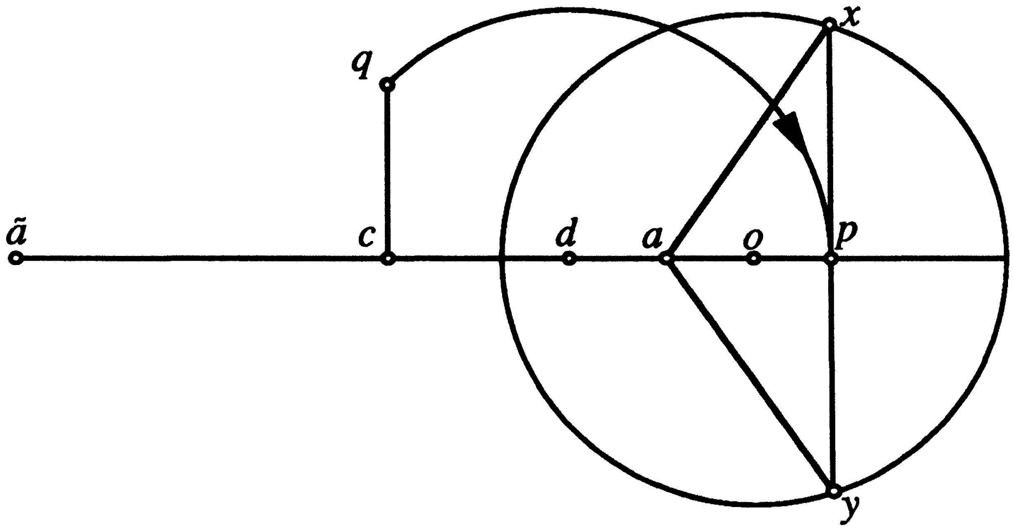

Fig. 4

Note that the general problem given in the introduction does not allow a constructive solution.

References

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