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The Problem of the Circular Billiard

Jörg Waldvogel, ETH Zürich

Jörg Waldvogel studierte an der ETH Zürich. Nach seiner Promotion zum Dr. sc. math. im Jahre 1966 war er längere Zeit in den USA tätig, zuerst von 1967 bis 1970 als Research Scientist bei Lockheed Missiles and Space Company in Huntsville, Alabama und als part-time Assistant Professor an der University of Alabama at Huntsville, und anschliessend von 1970 bis 1972 als Assistant Professor an der University of Texas at Austin. 1972 kehrte er an die ETH Zürich zurück; er wurde Lehrbeauftragter im Gebiete der numerischen und angewandten Mathematik und übernahm die Leitung der Numerikgruppe am Seminar für Angewandte Mathematik. 1985 erfolgte seine Ernennung zum Titularprofessor. Längere Aufenthalte verbrachte er als Gastprofessor an der Université de Paris VI (1980) und als Visiting Professor an der University of South Florida in Tampa (1986). Seine Forschungsinteressen liegen in den Gebieten Differentialgleichungen, Asymptotik, Geometrie und numerische Algorithmen.

Das Billardspiel übt auf viele Mathematiker (und Physiker) allein schon deswegen eine grosse Faszination aus, weil es erlaubt, die — wenigstens im idealisierten Fall — überschaubaren Stoss- und Reflexionsgesetze der Billardkugel im Experiment zu testen. Dass sich der Mathematiker dann auch theoretisch mit diesem Spiel und den Bewegungen der Billardkugel auseinandersetzt, liegt fast auf der Hand: eine sprudelnde Quelle von reizvollen mathematischen Problemen ist damit entdeckt. Unter diesen Problemen befinden sich ganz leichte und auch ganz schwierige, sogar solche, die bis heute ungelöst sind. Obschon an sich geometrischer Art, führen sie öfter zu Fragen, die auch andere mathematische Gebiete betreffen, und vielfach werden dann unerwartete Verbindungen sichtbar. — Jörg Waldvogel betrachtet in seinem Beitrag die Bewegungen einer (idealisierten) Billardkugel in einem Kreis. Die direkte rechnerische Behandlung dieses scheinbar einfachen Problems führt zu langen komplizierten Formeln. Jörg Waldvogel zeigt, wie die Einführung von komplexen Zahlen, verbunden mit geschickten geometrischen Überlegungen, eine kurze und elegante Lösung liefert. Dabei werden auch überraschende Zusammenhänge mit der Geometrie von gewissen Kurven in der Ebene aufgedeckt. Im anschliessenden Beitrag von Norbert Hungerbühler wird das gleiche Billardproblem auf ebenfalls sehr elegante Weise mit Hilfe von rein geometrischen Argumenten behandelt. *ust*

In the mathematical literature various kinds of billiard problems have been considered (see, e.g., [6]). The common feature is a billiard table in the shape of a given planar region with an ideally reflecting boundary and a point-like billiard ball moving without friction. Many interesting questions about the path of the bouncing billiard ball have been discussed. Here we will be concerned with the problem of finding all possible paths connecting two given points A, B on a circular billiard table of radius 1 via a single bounce at the unknown point Z .

In a first approach [2] one is tempted to introduce the polar angle of Z as an unknown and to set up the condition of equality of the reflexion angles at Z by means of trigonometry. This approach, however, quickly leads to unwieldy equations, and in [2] it is suggested to use computer algebra [1] in order to facilitate their simplification.

In this note we suggest to use complex notation for setting up the condition on the reflexion point Z and for simplifying it.

Let therefore two points $a, b \in \mathbb{C}$, $0 < |a| < 1$, $0 < |b| < 1$ be given in the punctuated unit disk by their respective complex coordinates. The problem is to find all reflexion points $\zeta_0 = e^{i\varphi}$ on the unit circle such that the segments $\zeta_0 - a$ and $b - \zeta_0$ form the path of an elastically bouncing billiard ball. We will present a short and elegant geometric-algebraic method of deriving and discussing the condition to be satisfied by the reflexion point ζ_0 .

1 Geometry

We define L as the locus of all points $z \in L \subset \mathbb{C}$ such that the angles $\sphericalangle(a, z, 0)$ and $\sphericalangle(0, z, b)$ are equal (see Figure 1). Obviously, the reflexion points ζ_0 are the points of intersection of L with the unit circle. L is given by the condition

$$\arg\left(\frac{z-a}{z}\right) = \arg\left(\frac{z}{z-b}\right) \quad \text{or} \quad \arg\left(\frac{(z-a)(z-b)}{z^2}\right) = 0,$$

or equivalently

$$(1) \quad \frac{(z-a)(z-b)}{z^2} = r > 0$$

where r is a real parameter. To find the coordinate equation of L we eliminate r by combining Equ. (1) with its complex conjugate:

$$(2) \quad r - 1 = \frac{a b}{z^2} - \frac{a + b}{z} = \frac{\bar{a} \bar{b}}{\bar{z}^2} - \frac{\bar{a} + \bar{b}}{\bar{z}}.$$

By introducing the mirror image $\zeta = 1/\bar{z}$ of z with respect to the unit circle Equ. (2) is transformed into the quadratic relation

$$(3) \quad \bar{a} \bar{b} \zeta^2 - a b \bar{\zeta}^2 - (\bar{a} + \bar{b}) \zeta + (a + b) \bar{\zeta} = 0.$$

Therefore the mirror image Λ of the locus L is a conic section: it is easily seen that Λ is an equilateral hyperbola with asymptotes parallel to the directions $\zeta = \sqrt{\pm a b}$, i.e.

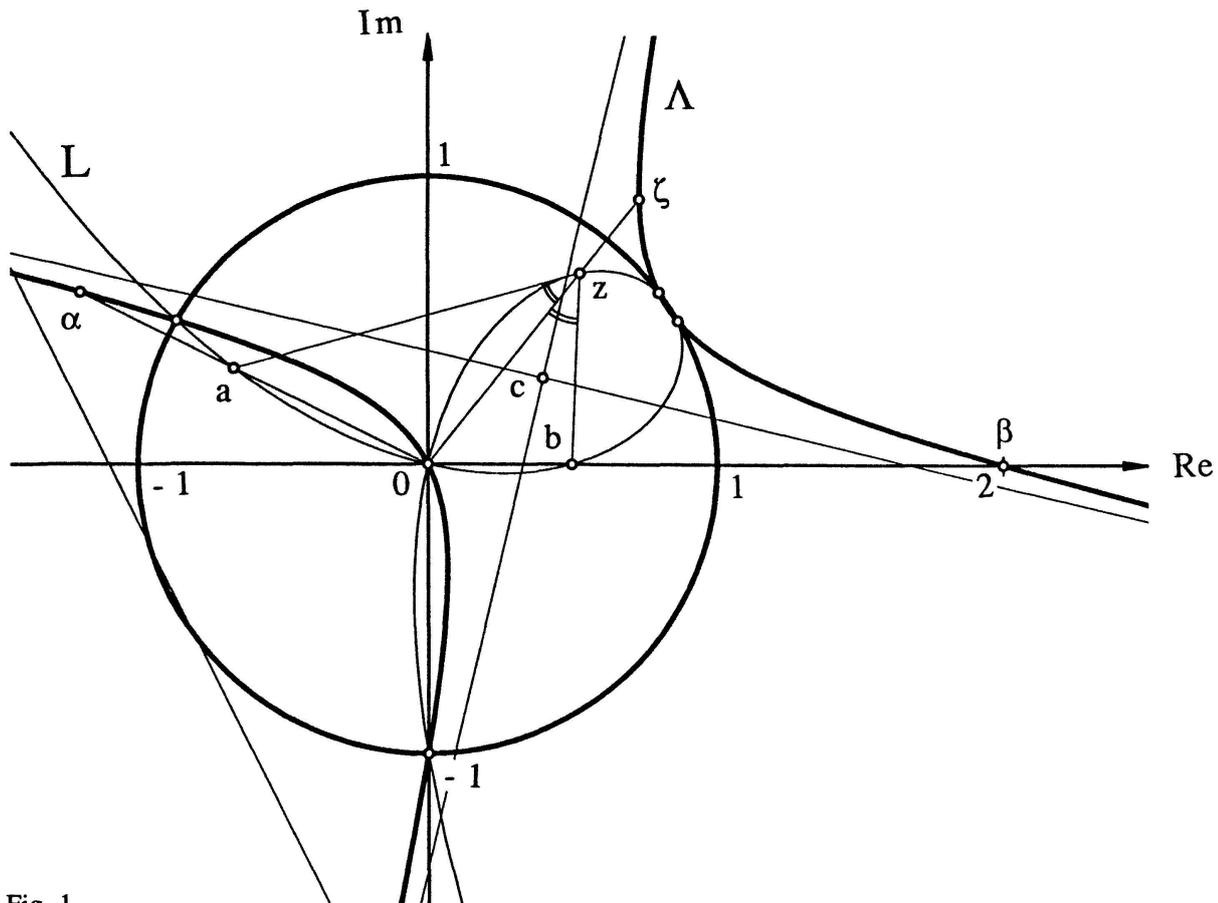


Fig. 1

parallel to the two (orthogonal) bisectors of the angle $\sphericalangle(a, 0, b)$. The center c of the hyperbola is

$$(4) \quad c = \frac{1}{2} \frac{\bar{a} + \bar{b}}{\bar{a} \bar{b}} = \frac{1}{2} \left(\frac{1}{\bar{a}} + \frac{1}{\bar{b}} \right) = \frac{1}{2} (\alpha + \beta),$$

i.e. c is the midpoint of the segment between the mirror images $\alpha = 1/\bar{a}$, $\beta = 1/\bar{b}$ of a, b . The hyperbola Λ passes through the points $\zeta = 0$, $\zeta = \alpha$, $\zeta = \beta$.

A geometrical proof of the theorem that the mirror image Λ of L is an equilateral hyperbola has been given by N. Hungerbühler [3]. In contrast to the simple nature of Λ , the locus L is an algebraic curve of degree 3, as is seen by multiplying Equ. (2) by $z^2 \bar{z}^2$ (see Figure 1).

Since the unit circle is pointwise invariant under the inversion $z \rightarrow \zeta = 1/\bar{z}$ the reflexion points ζ_0 are the points of intersection of the hyperbola Λ with the unit circle

$$(5) \quad \zeta \bar{\zeta} = 1 \quad .$$

2 Algebra

The reflexion points ζ_0 satisfy both Eqs. (3) and (5). Multiplying (3) by ζ^2 and using (5) yields the single 4th degree equation

$$(6) \quad q(\zeta) := \bar{a} \bar{b} \zeta^4 - (\bar{a} + \bar{b}) \zeta^3 + (a + b) \zeta - a b = 0$$

satisfied by ζ_0 . The same equation is satisfied by $1/\bar{\zeta}_0$; therefore the solutions of (6) occur either in inverse pairs, or they lie on the unit circle; only the latter ones are solutions of the billiard problem.

To write Equ. (6) in real form we introduce the new variable t according to

$$(7) \quad \zeta = e^{i\varphi} = \frac{1 + i t}{1 - i t}, \quad t = \tan \frac{\varphi}{2},$$

where φ is the polar angle of ζ ; furthermore we introduce the abbreviations

$$(8) \quad s = s_1 + i s_2 = a + b, \quad p = p_1 + i p_2 = a b$$

for the sum and product of a and b , where s_1, p_1, s_2, p_2 are the respective real and imaginary parts. Then we obtain

$$q(\zeta) = -2i[(t^4 - 6t^2 + 1)p_2 + 4(t^3 - t)p_1 + (t^2 + 1)(2t s_1 + (t^2 - 1) s_2)](1 - i t)^{-4},$$

and Equ. (6) becomes

$$(9) \quad (s_2 + p_2) t^4 + 4\left(\frac{s_1}{2} + p_1\right) t^3 - 6p_2 t^2 + 4\left(\frac{s_1}{2} - p_1\right) t + (-s_2 + p_2) = 0.$$

As a consequence of Section 1 it has always 2 or 4 real solutions t_k , directly connected with the arguments φ_k of the reflexion points by means of Equ. (7).

3 Example

To illustrate the computations involved we consider the example

$$(10) \quad a = \frac{-2 + i}{3}, \quad b = \frac{1}{2}.$$

According to (4) the center of the hyperbola Λ is $c = 0.4 + 0.3i$; the geometric situation is shown in Figure 2. Incidentally, the polynomial $q(\zeta)$ in Equ. (6) may easily be factored as

$$6q(\zeta) = (-2 - i) \zeta^4 + (1 + 2i) \zeta^3 + (-1 + 2i) \zeta + 2 - i = (\zeta^3 - i)[(-2 - i) \zeta + (1 + 2i)] = 0;$$

therefore we immediately obtain the following 4 solutions of the billiard problem:

$$(11) \quad \zeta_1 = -i, \quad \zeta_{2,3} = \frac{1}{2}(\mp\sqrt{3} + i), \quad \zeta_4 = 0.8 + 0.6i.$$

Alternatively, Equ. (9) (after multiplication by 6) becomes

$$3t^4 - 10t^3 - 6t^2 + 6t - 1 = (t + 1)(t^2 - 4t + 1)(3t - 1) = 0$$

with the solutions

$$(12) \quad t_1 = -1, \quad t_{2,3} = 2 \pm \sqrt{3}, \quad t_4 = \frac{1}{3},$$

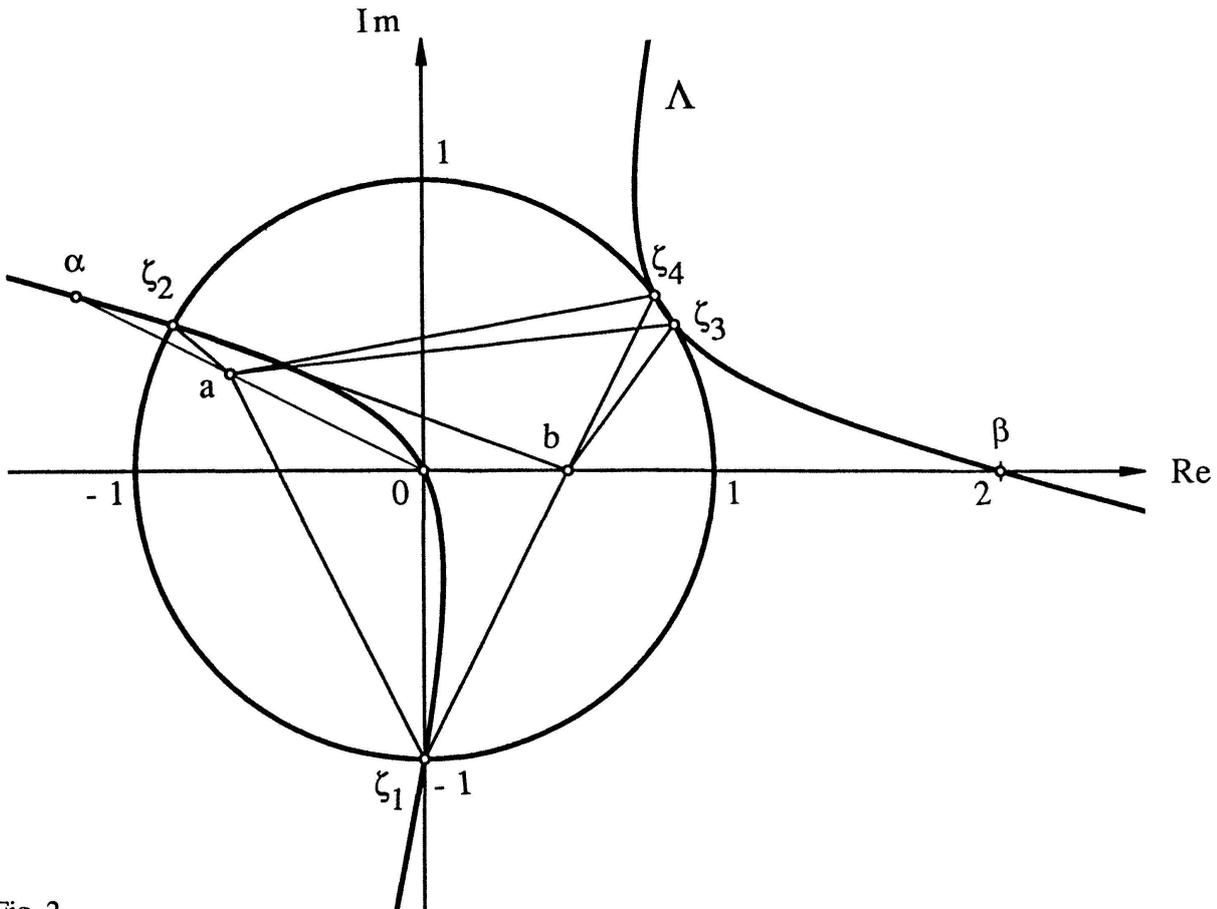


Fig. 2

and, according to (7), the polar angles

$$\varphi_1 = -\frac{\pi}{2}, \quad \varphi_2 = \frac{5\pi}{6}, \quad \varphi_3 = \frac{\pi}{6}, \quad \varphi_4 = \arctan\left(\frac{3}{4}\right) = 0.6435011088,$$

in perfect agreement with the complex values (11).

In general the 4th degree equations (6) or (9) must be solved by means of a polynomial root finder. In the currently available mathematical software, e.g. [1,4,7], such algorithms are readily available. Mostly, reliable algorithms like the QR transformation for solving eigenvalue problems, tuned to high perfection, are implemented in these systems. Equ. (9) is more efficient than Equ. (6) since it avoids complex arithmetic.

For the casual programmer the Newton-Raphson algorithm (see, e.g., [5]) is an adequate numerical method: To find a zero ζ_0 of the function $q(\zeta)$, choose a complex initial approximation $\zeta^{(0)}$ sufficiently close to ζ_0 and iterate

$$\zeta^{(k+1)} = \zeta^{(k)} - \frac{q(\zeta^{(k)})}{q'(\zeta^{(k)})}, \quad k = 0, 1, \dots,$$

where $q'(\zeta)$ is the derivative of $q(\zeta)$; then $\zeta_0 = \lim_{k \rightarrow \infty} \zeta^{(k)}$. The above example may serve as a test case.

4 Double Roots

The closeness of the reflexion points ζ_3, ζ_4 in this example calls for establishing the condition for the existence of a double root in the polynomial $q(\zeta)$. Many of the currently available symbolic computation systems, e.g. [1,7], allow to handle this otherwise cumbersome task by means of the discriminant $\Delta(s, p)$ of the polynomial $q(\zeta)$ (see Eqs. (6), (8)). The condition for the existence of a multiple root, written as $\Delta(s, p) = 0$, is obtained by eliminating ζ from the simultaneous conditions $q(\zeta) = 0$, $q'(\zeta) = 0$. The result is

$$(13) \quad \Delta(s, p) = 4|s|^6 + 6|s|^4|p|^2 - 54\operatorname{Re}(\bar{s}^4 p^2) + 192|s|^2|p|^4 - 256|p|^6,$$

whereas the discriminant of the polynomial in Equ. (9) is found to be $(-2i)^6 \cdot \Delta(s, p)$. If $\Delta < 0$, $q(\zeta)$ has 4 distinct zeros on the unit circle. In the above example the discriminant has the rather small value $\Delta = -1/108$.

References

- [1] Char, B.W., Geddes, K.O., Gonnet, G.H., Leong, B.L., Monagan, M.B. and Watt, S.M., 1991: *Maple V Language/Reference Manual*, Springer-Verlag, New York.
- [2] Gander, W. and Gruntz, D., 1991: *The Billiard Problem*. Submitted to the International Journal of Mathematical Education in Science and Technology.
- [3] Hungerbühler, N., 1991: *Private communication*.
- [4] *Matlab User's Guide*, 1989. The Math Works Inc., South Natick, Montana.
- [5] Schwarz, H.R., 1989: *Numerical Analysis*. John Wiley & Sons Ltd.
- [6] Sinai, Ya.G. (Ed.), 1989: *Encyclopaedia of Mathematical Sciences*, vol. 2, *Dynamical Systems II*. In particular p. 151–172. Springer-Verlag.
- [7] Wolfram, S., 1988: *Mathematica*. Addison-Wesley Publishing Co.

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