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A Useful Banach Algebra

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Wolfgang Walter studierte in Tübingen und promovierte 1956 bei E. Kamke. Seit 1963 lehrt er an der Universität Karlsruhe als Professor für Mathematik, unterbrochen durch Gastaufenthalte in Amerika. Sein Hauptarbeitsgebiet ist die Theorie der Differentialgleichungen mit dem Schwerpunkt bei den nichtlinearen Problemen und bei Differential-Ungleichungen. Er ist der Verfasser von vielen wissenschaftlichen Arbeiten und von mehreren Lehrbüchern und Monographien.

In this note we introduce a Banach algebra of power series and give two applications, (i) an existence proof for the initial value problem $w'(z) = f(z, w(z))$, $w(z_0) = w_0$, and (ii) a proof of the implicit function theorem. These proofs are short, and they use only elementary results about power series, but no theorems on holomorphic functions. If only real power series are considered, one obtains "real" proofs for theorems of real analysis. Regarding problem (ii), most books (and courses) on real analysis contain the theorem that under the assumptions $f \in C^k$, $f(0,0) = 0$, $f_y(0,0) \neq 0$ the equation $f(x,y) = 0$ defines a function $y = \varphi(x) \in C^k$ near $x = 0$. Important examples (and most exercises) are of the form $f(x,y) = \sum a_{ij}x^iy^i$; yet the theorem that in this case φ has a power

Abstraktion ist in der Mathematik kein Selbstzweck. Es besteht kein Zweifel, dass abstrakte Begriffe die weitere Entwicklung eines Gebietes nur dann überdauern, wenn sie zu vermehrter Einsicht in ein Problem oder sogar zur Lösung eines Problems führen. Leider geht dieser Aspekt in der Darstellung der Mathematik, insbesondere in der Lehre, manchmal verloren, und die Studentinnen und Studenten vermögen vor lauter abstrakten Begriffen das konkrete interessante Problem nicht mehr zu sehen. In seinem Artikel gibt Wolfgang Walter ein Beispiel dafür, wie ein abstrakter Begriff der Analysis zur Lösung eines konkreten und klassischen Problems beitragen kann, obschon er dafür ursprünglich gar nicht geschaffen wurde. Der Begriff der Banachalgebra wird eingesetzt, um zwei grundlegende Sätze der reellen bzw. komplexen Analysis zu beweisen, nämlich den Existenz- und Eindeutigkeitssatz für das Anfangswertproblem von Differentialgleichungen erster Ordnung und den Satz über implizite Funktionen. Beide Sätze verlangen in der klassischen Darstellung umständliche Beweise. Walter zeigt, wie sie elegant und auf kurzem Weg zu erhalten sind, wenn der Begriff der Banachalgebra herangezogen wird. ust

series expansion is lacking. A proof is given in Section 4; the author is not aware of other real proofs of that theorem.

1 The Banach algebra H_r

Let $H_r (r > 0)$ be the vector space of all functions u allowing a power series expansion $u(z) = \sum_{k=0}^{\infty} c_k z^k (c_k \in \mathbb{C})$ which is absolutely convergent for $z = r$, and let

$$\|u\| = \sum_{k=0}^{\infty} |c_k| r^k < \infty.$$

Obviously, the power series is uniformly convergent in the closed disk $|z| \leq r$, and $|u(z)| \leq \|u\|$ in that disk.

Theorem 1. *H_r is a Banach algebra, i.e., a Banach space and an algebra such that* $\|uv\| \leq \|u\| \, \|v\|$.

Proof. It is easily seen that H_r is a vector space and that $\|\cdot\|$ is a norm. For example, the triangle inequality reduces to $\sum |c_k + d_k| r^k \leq \sum |c_k| r^k + \sum |d_k| r^k$. To prove completeness, we consider a Cauchy sequence (u_n), where $u_n = \sum_k c_k^n z^k$. Since

$$|c_k^m - c_k^n| r^k \leq \|u_m - u_n\| \to 0 \quad \text{as} \quad m, n \to \infty,$$

$(c_k^n)_{n=1}^{\infty}$ is a Cauchy sequence of complex numbers, and $\lim_{n \to \infty} c_k^n = d_k$ exists. Let $\varepsilon > 0$ be given. Then there exists n_0 such that for any finite sum $\sum'_k$

$$\sum_k{}' |c_k^m - c_k^n| r^k \leq \|u_m - u_n\| < \varepsilon \quad \text{for} \quad m, n > n_0$$

and hence

$$\sum_k{}' |c_k^m - d_k| r^k \leq \varepsilon \quad \text{for} \quad m > n_0.$$

The same inequality holds for the infinite sum where k runs from 0 to ∞. The estimate $|d_k| \leq |c_k^m| + |c_k^m - d_k|$ then shows that $v(z) = \sum d_k z^k$ belongs to H_r, and the inequality $\|u_m - v\| \leq \varepsilon$ indicates that $\lim u_m = v$ in H_r.

Now let $u = \sum c_k z^k$, $v = \sum d_k z^k \in H_r$ and $w = uv$. Taking Cauchy products, one gets

$$\|w\| = \sum_k r^k \left| \sum_{i+j=k} c_i d_j \right| \leq \sum_k r^k \sum_{i+j=k} |c_i| \, |d_j| = \|u\| \cdot \|v\|. \qquad \square$$

The following simple facts are given without proof.

(a) $\|z^k\| = r^k$ $(k = 0, 1, \ldots)$, in particular $\|1\| = 1$.

(b) $u \in H_r$ implies $u^k \in H_r$ and $\|u^k\| \leq \|u\|^k$ $(k = 0, 1, 2, \ldots)$.

(c) If (u_n) is a sequence in H_r such that $\sum \|u_n\| < \infty$, then $u = \sum u_n \in H_r$ and $\|u\| \leq \sum \|u_n\|$.

(d) The integration operator I,

$$(Iu)(z) = \sum_{k=0}^{\infty} c_k \frac{z^{k+1}}{k+1} \quad \text{where} \quad u = \sum_{k=0}^{\infty} c_k z^k,$$

maps H_r into itself, and $\|Iu\| \leq r \|u\|$ with equality for $u = 1$, hence $\|I\| = r$.

Remarks. 1. H_r as defined above is a complex Banach algebra. If we consider only real power series $u(x) = \sum c_k x^k (c_k \in \mathbb{R})$, the corresponding real Banach algebra is obtained. 2. A similar theorem holds for multiple power series $u(z) = \sum c_p z^p$, where $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$, $p = (p_1, \ldots, p_n)$ with $p_i \geq 0$. Let $r = (r_1, \ldots, r_n)$ with $r_i > 0$ and define $\|u\| = \sum c_p r^p$. Then the set H_r of all power series with finite norm is again a real or complex Banach algebra. The condition $\|u\| < \infty$ implies that the power series is uniformly absolutely convergent in the set $|z_i| \leq r_i$ $(i = 1, \ldots, n)$.

2 The initial value problem

Consider the problem $w' = f(z, w)$, $w(z_0) = w_0$. By introducing $z - z_0$ as a new independent variable and $w - w_0$ as a new dependent variable, the problem is reduced to the case where $z_0 = 0$, $w_0 = 0$. For reasons of simple notation we deal with this case only

$$w'(z) = f(z, w(z)), \quad w(0) = 0. \tag{1}$$

Theorem 2. *Let* $f(z, w) = \sum a_{ij} z^i w^j$ *(here and below,* $i, j \geq 0$*) and assume that, for positive numbers* r, b,

$$M = \sum |a_{ij}| r^i b^j < \infty \quad \textit{and} \quad L = \sum |a_{ij}| r^i j b^{j-1} < \infty.$$

Then, if $r \leq b/M$ *and* $r < 1/L$, *the initial value problem (1) has a unique solution in* H_r.

Proof. Equation (1) is equivalent to the fixed point equation

$$w = IFw, \quad \text{where} \quad (Fw)(z) := f(z, w(z)), \tag{2}$$

and I is the integration operator defined in (d).
We show that the operator IFw maps the closed ball B: $\|u\| \leq b$ in H_r into itself, and that it is a contraction. Let $u \in B$. The function $v_k = \sum^{(k)} a_{ij} z^i u^j$, where summation runs over all indices i, j with $i + j = k$, belongs to H_r, and from (a) and (b) the estimate $\|v_k\| \leq \sum^{(k)} |a_{ij}| r^i b^j$ follows. Since $\sum \|v_k\| \leq M$, it follows from (c) that $v = \sum v_k = Fu \in H_r$ and $\|Fu\| \leq M$. Hence $\|IFu\| \leq rM \leq b$ by (d). This shows that $IF(B) \subset B$.
Now let $\|u\|, \|v\| \leq b$. The decomposition

$$u^j - v^j = (u - v)(u^{j-1} + u^{j-2} v + \ldots + v^{j-1})$$

implies that $\|u^j - v^j\| \leq \|u - v\| j b^{j-1}$ and hence

$$\|Fu - Fv\| \leq \sum |a_{ij}| \, \|z^i\| \, \|u^j - v^j\| \leq \sum |a_{ij}| r^i j b^{j-1} \|u - v\| = L\|u - v\|.$$

According to (d) we have $\|IFu - IFv\| \leq rL\|u - v\|$ with $rL < 1$, i.e., the operator IF is a contraction. The theorem now follows from the contraction principle. □

Remarks.

1. The proof shows only that there is a unique solution, say w, in H_r. Is there another power series solution u which does not belong to H_r? If yes, then, for some positive numbers $s \le r$, u and w belong to H_s, and the above proof (performed in H_s) shows that $u = w$.

2. If all coefficients a_{ij} of f are real, one takes the real Banach algebra H_r and obtains a real power series solution. This proof uses no knowledge on complex numbers or functions whatsoever.

3. The constant M is an upper bound for $|f|$ in the region $|z| \le r$, $|w| < b$, and the restriction $r \le b/M$ is familiar from existence theory. But there is another condition $r < 1/L$ which is not found in other approaches (we remark that L is an upper bound for $|\partial f/\partial w|$). One can get rid of it in the following way. Let $S_n u = u_0 + u_1 z + \ldots + u_n z^n$ be a partial sum and $R_n u = u - S_n u$ the corresponding rest of the power series. If condition $r < 1/L$ is not satisfied, we choose s such that $0 < s < 1/L \le r$. Theorem 2 gives a solution $\bar{w} \in H_s$. Since $\bar{w} = IF\bar{w}$ implies $S_n \bar{w} = S_n IF\bar{w}$, the solution $\bar{w}$ satisfies the equation $\bar{w} = S_n \bar{w} + R_n IF\bar{w}$, i.e., $\bar{w}$ is a solution to the equation

$$w = g + R_n IFw, \quad \text{where} \quad g = S_n \bar{w}. \qquad (*)$$

It is easily seen that $R_n I$ is a linear operator in H_r with norm $||R_n I|| = r/(n+1)$. It follows that the operator $R_n IF$ satisfies a Lipschitz condition with constant $rL/(n+1)$. We choose n such that $rL < n+1$, by applying the contraction principle, obtain a solution of equation $(*)$ which must be $\bar{w}$ (see Remark 1). Hence b/M is a lower bound for the radius of convergence of the solution.

3 System of differential equations

Now let $w' = f(z, w)$ denote a system of n differential equations in vector notation. We introduce the Banach space H_r^n of functions $w = (w_1, \ldots, w_n)$, where $w_1, w_2, \ldots$ belong to H_r, and use the maximum norm

$$||w||_n = \max\{||w_i|| : i = 1, \ldots, n\}.$$

In what follows, i runs from 1 to n, j from 0 to ∞, and $p = (p_1, \ldots, p_n)$ is a multi-index, $|p| = p_1 + \ldots + p_n$, $p_i \ge 0$.

(e) $u \in H_r$, $w \in H_r^n$ implies $uw = (uw_i) \in H_r^n$ and $||uw||_n \le ||u||\ ||w||_n$.

(f) For $w \in H_r^n$, $w^p = w_1^{p_1} \ldots w_n^{p_n} \in H_r$ satisfies $||w^p|| \le ||w||_n^{|p|}$.

(g) $v, w \in H_r^n$, $||v||_n, ||w||_n \le s$ implies $||v^p - w^p|| \le ||v - w||_n |p| s^{|p|-1}$.

We indicate the proof of (g), using the example $n = 3$, $p = (1, 2, 4)$. With $v = (a, b, c)$, $w = (\bar{a}, \bar{b}, \bar{c})$ one gets

$$\begin{aligned} v^p - w^p &= (a - \bar{a})b^2c^4 + \bar{a}(b^2 - \bar{b}^2)c^4 + \bar{a}\bar{b}^2(c^4 - \bar{c}^4) \\ &= (a - \bar{a})b^2c^4 + (b - \bar{b})(b + \bar{b})\bar{a}c^4 \\ &\quad + (c - \bar{c})(c^3 + c^2\bar{c} + c\bar{c}^2 + \bar{c}^3)\bar{a}\bar{b}^2 \end{aligned}$$

and

$$||v^p - w^p|| \le s^6||a - \bar{a}|| + 2s^6||b - \bar{b}|| + 4s^6||c - \bar{c}|| \le 7s^6||v - w||_n.$$

With these tools it is easy to prove the following

Theorem 3. *Let* $w = (w_1, \ldots, w_n)$, $f(z,w) = (f_1, \ldots, f_n) = \sum a_{jp} z^j w^p$ *with* $a_{jp} \in \mathbb{C}^n$ *and assume that* $r, b > 0$ *exist such that*

$$M = \sum |a_{jp}| r^j b^{|p|} < \infty,$$

where $|\cdot|$ *denotes the maximum norm in* $\mathbb{C}^n$. *Then problem (1) has a unique power series solution which converges (at least) in the circle* $|z| < e = \min(r, b/M)$.

The *proof* runs basically as before. Let $0 < c < b$ and $s = \min(r, c/M)$. Then

$$L = \sum |a_{jp}| s^j |p| c^{|p|-1} < \infty,$$

since $|p| c^{|p|-1} < b^{|p|}$ for large $|p|$. We consider equation (2). Using (e)–(g) it is shown as before that the operator IF maps the closed ball $B = \{u \in H_s^n : \|u\|_n \le c\}$ into itself, and that for $u, v \in B$ a Lipschitz condition $\|Fu - Fv\|_n \le L\|u - v\|_n$ holds. If s satisfies also $sL < 1$, then IF is a contraction, and the contraction principle guarantees that a unique solution $w \in H_s^n$ exists. This additional restriction is removed as in the preceding Remark 3. Since c can be chosen arbitrarily close to b, the theorem follows. □

4 Implicitly defined functions

Let

$$f(x,y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j, \quad f(0,0) = a_{00} = 0, \quad f_y(0,0) = a_{01} \neq 0, \tag{3}$$

where $a_{ij}, x, y \in \mathbb{R}$. We are going to prove the following

Implicit function theorem. *Assume that (3) holds, the series being absolutely convergent for* $|x| \le a$, $|y| \le b$ $(a, b > 0)$. *Then there are positive numbers* $r \le a$, $s \le b$ *and a function* $w \in H_r$ *such that* $f(x, w(x)) = 0$ *for* $|x| \le r$, $|w(x)| \le s$ *and* $f(x,y) \neq 0$ *for all points* $(x,y) \in [-r,r] \times [-s,s]$ *different from* $(x, w(x))$.

Proof. Equation $f(x,y) = 0$ is equivalent to

$$y = \sum_{i,j=0}^{\infty} b_{ij} x^i y^j =: g(x,y),$$

where $b_{00} = b_{01} = 0$ and $b_{ij} = -a_{ij}/a_{01}$ otherwise. We use the notation $Gw = g(x, w(x))$ and solve the fixed point equation $w = Gw$ by reduction to the contraction principle in the Banach algebra H_r. The numbers r, s are chosen in such a way that

$$B = \sum_i |b_{i0}| r^i \le \frac{1}{2} s \quad \text{and} \quad L = \sum_{i,j} |b_{ij}| r^i j s^{j-1} \le \frac{1}{2}.$$

It is seen as in the proof of Theorem 2 that $\|u\|, \|v\| \le s$ implies $\|Gu - Gv\| \le L\|u - v\| \le \frac{1}{2}\|u - v\|$. Furthermore, $B = \|G(0)\|$, and the estimate $\|G(u)\| \le \|G(0)\| + \|G(u) - G(0)\| \le$

$\frac{1}{2}s + L\|u\| \le s$, valid for all $u \in H_r$ satisfying $\|u\| \le s$, shows that G maps the closed ball $\|u\| \le s$ into itself. By the contraction principle, a fixed point $\bar{w}$ of G exists, and it is unique in H_r. □

In order to show that $y \neq \bar{w}(x)$ implies $f(x,y) \neq 0$, one may use the Banach algebra of bounded functions $u : [-r,r] \to \mathbb{R}$ with norm $\|w\| = \sup\{|w(x)| : |x| \le r\}$. The above proof works also in this norm. It shows that $\bar{w}$ is the only bounded function satisfying $|\bar{w}(x)| \le s$ and $f(x,\bar{w}(x)) = 0$ in $[-r,r]$.

Remark. The theorem is also true in the complex case, i.e., where $a_{ij}, x, y \in \mathbb{C}$. Moreover it remains true in the case where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $f \in \mathbb{R}^m$, i.e., $f(x,y) = \sum a_{pq}x^p y^q$ with $a_{pq} \in \mathbb{R}^m$, $p = (p_1,\dots,p_n)$, $q = (q_1,\dots,q_m)$. It is assumed that $a_{00} = f(0,0) = 0$ and $A = \frac{\partial f}{\partial y}(0,0)$ is an invertible $m\times m$ matrix. The equation $f = 0$ can be written as $Ay = -\sum' a_{pq}x^p y^q = 0$, where summation in $\sum'$ is taken over all (p,q) except $(0,q)$ with $|q| = 1$. The equivalent fixed point equation $y = -\sum'(A^{-1}a_{pq})x^p y^q =: g(x,y)$ can be treated as before.

Closing remark. The Banach algebra H_r was used by Harris, Sibuya and Weinberg (1969) in their simplified treatment of complex linear differential systems at singular points. In his book (1990, first edition 1972) the present author used this approach and extendend it to the case where solutions with logarithmic terms appear. The present note shows that it can be employed effectively in real nonlinear problems.

The algebra H_r can be used for a short and elegant proof of two fundamental theorems in the theory of functions of several complex variables, the Preparation Theorem and the Division Theorem of Weierstraß. This proof has been propagated by Grauert and Remmert since the sixties and is found, e.g., in their books (1971) and (1984), p. 39–42 (oral communication by Prof. Remmert). In this light the above proof of the implicit function theorem appears as a preparation to the Preparation Theorem.

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