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Dissections into equilateral triangles

Abstract. In a recent paper [El. Math., Vol. 46/4] H. Kaiser proved that every non-equilateral triangle has a dissection into finitely many (at most eight) similar but pairwise incongruent triangles. In this note we show that an equilateral triangle – and, more generally, any convex polygon – has no finite dissection into incongruent equilateral triangles. We also prove that the maximum number $f(n)$ of distinct sizes of equilateral triangles in a dissection into n parts is equal to $cn - o(n)$ for some constant $c \geq 5/7$.

1. Introduction

Let Π and Π' be polygons in the Euclidean plane. A dissection of Π into Π' is a decomposition of Π into finitely many, internally disjoint polygons $\Pi'_1, \ldots, \Pi'_n$ ($n \geq 2$) such that all of the Π'_i are similar to Π'. A dissection is perfect if the Π'_i are pairwise incongruent. Those Π'_i will be called the tiles of the dissection. The symbol Δ will be a shorthand for «equilateral triangle» (possibly with distinct side lengths at different places).

In the past fifty years it was extensively studied how squares and rectangles can be dissected into smaller squares. A detailed account on the history of this problem, with numerous references, can be found in the survey [2].

Relatively little is known, however, about dissections of polygons other than the square. Tutte [4, § 2] proved that a Δ has no perfect dissection into smaller Δ's (this result was stated without proof in the classic paper [1]), and quite recently Kaiser [3] observed that in fact Δ is the only «exceptional case», i.e. every non-equilateral triangle has a perfect dissection (into at most eight tiles).

In this note, applying an argument much shorter than the original one in [4], we prove the following extension of Tutte's theorem.

Theorem 1. *Every dissection of a convex polygon into equilateral triangles contains two triangles of the same size.*

Knowing that every finite dissection of Δ contains at least two congruent tiles, it is natural to raise the following problem.

Problem 1. *Given a positive integer $n \geq 6$, dermine the largest number $f(n)$ of distinct side lengths in a dissection $\Delta_1 \cup \ldots \cup \Delta_n = \Delta$ of an equilateral triangle Δ.*

Figure 1 shows that $f(n)$ is well-defined for every $n \geq 6$, and also that $f(n) \geq 2$. (The dissection into 4 tiles is unique, with $f(4) = 1$, and no dissection exists with 5 tiles.) Here we prove

Theorem 2. *There is a positive constant c, $5/7 \leq c \leq 1$, such that $f(n) = cn - o(n)$ as $n \to \infty$.*

It would be interesting to determine the exact value of $\lim_{n \to \infty} f(n)/n$. In particular, decide whether or not the limit is less than 1.

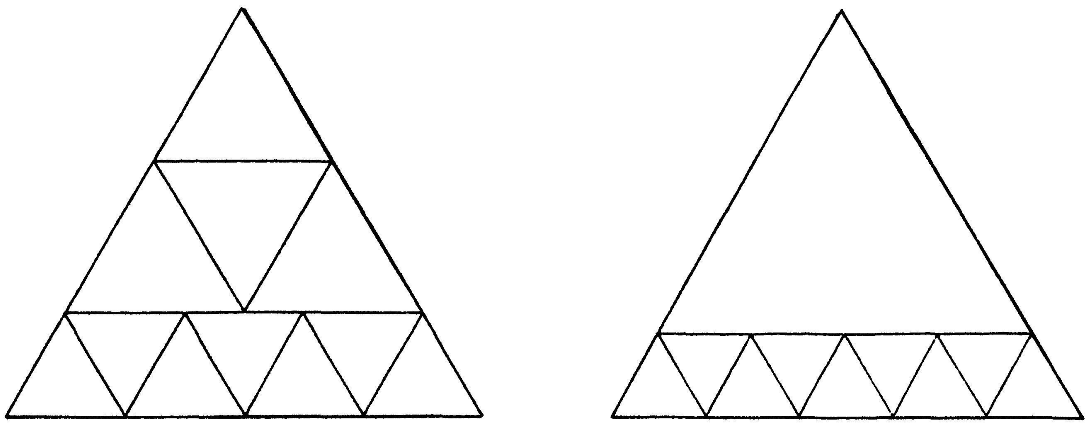

Figure 1. Dissection into an odd or even number of triangles.

We note that Δ has a «perfect dissection» into countably many tiles whose total area is equal to the area of Δ. The idea to show this fact is that a large part of a «long» trapezoid can be filled with finitely many Δ's of nearly equal (but distinct) side lengths. It remains an open problem, however, to prove or disprove that no dissection into pairwise incongruent Δ's is locally finite apart from the neighborhoods of a finite number of points.

2. Proofs

Proof of Theorem 1.

Consider a dissection $\Delta_1 \cup \ldots \cup \Delta_n = \Pi$ of a convex polygon Π into $n \geq 2$ internally disjoint Δ's. We say that a Δ_i is split if a vertex of a Δ_j is an internal point of some side of Δ_i. If none of the Δ_i is split, then the (one, two, or three) neighbors of each Δ_i have the same size as Δ_i itself, and the connectedness of the dissection implies that all Δ_i are congruent. Hence, from now on we assume that there is at least one split triangle. Note that, by the assumption on convexity, each internal point splitting a side of some Δ_i is the vertex of precisely three tiles.

Suppose that $\Delta_1 = ABC$ is the smallest split triangle. We prove that an internal point of some side of Δ_1 is the vertex of two triangles Δ_i, $\Delta_{i'}$ of the same size.

A triangle Δ_j is said to overhang Δ_1 at A if the boundary of Δ_j entirely contains the side AB or AC of Δ_1, and A is not a vertex of Δ_j. Certainly, at each vertex of a triangle at most one neighboring triangle can overhang, and if Δ_i overhangs Δ_j then $|\Delta_i| > |\Delta_j|$ (where $|\Delta_i|$ denotes any measure – side length, area, etc.).

We find two Δ_i of the same size, depending on the distribution of vertices on the periphery of Δ_1. In the first three of the four possible cases we assume that each side of Δ_1 contains at most one internal vertex. In order to simplify some technical details of the argument, and to insure that each tile be surrounded completely by its neighbors, we artificially place an «external» Δ on each side of Π. Note that placing or removing external Δ's does not change the status of split triangles in the dissection.

Case 1: The periphery of Δ_1 contains just one internal vertex.

Say, D is an internal vertex on AB. The two neighbors of Δ_1 containing C cannot be smaller than Δ_1. Suppose that both of them are larger than Δ_1. Since at most one of them can overhang at C, the other must overhang at, say, A (see Fig. 2(a)). Then AD is the side of some triangle $\Delta_j = ADE$. Since $|\Delta_j| < |\Delta_1|$, Δ_j is not split by the choice of Δ_1. Moreover, at most one neighbor of Δ_j can overhang at E. Thus, AE or DE is the side of two triangles of the same size.

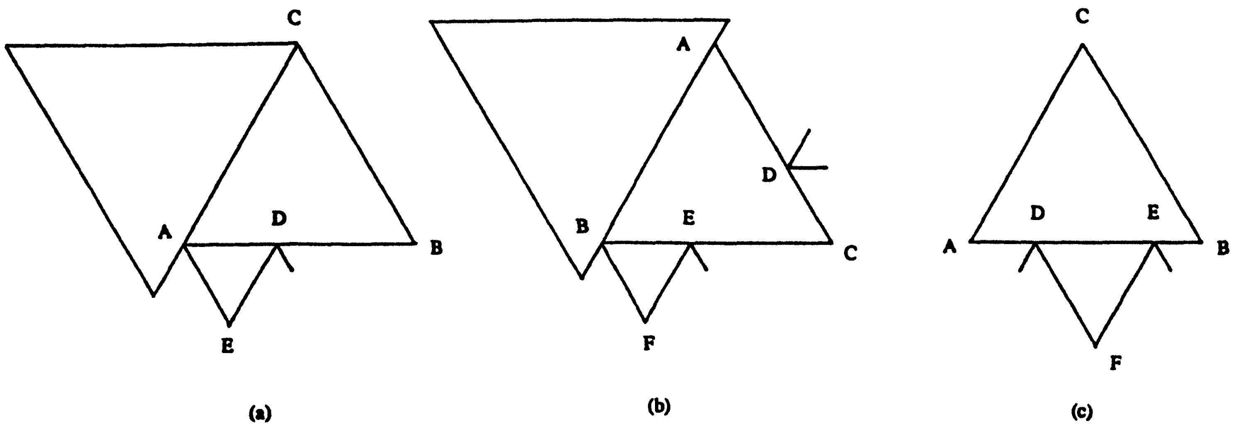

Figure 2.

Case 2: The periphery of Δ_1 contains precisely two internal vertices, on distinct sides.

Let D and E be internal vertices in AC and BC, respectively. Since the edge AB is not split, some neighbor Δ_i of Δ_1 entirely contains AB, and if $|\Delta_i| \neq |\Delta_1|$ then Δ_i overhangs Δ_1 at, say, B (see Fig. 2(b)). Then the triangle BEF is in a similar situation as ADE in Case 1, so that it has a neighbor of the same size.

Case 3: The periphery of Δ_1 contains precisely three internal vertices, on pairwise distinct sides.

Let D_i $(i = 1, 2, 3)$ be the internal vertices on the periphery of Δ_1. Consider the triangles whose boundaries contain the segments CD_1 and CD_2. Some of them does not contain C as an internal point of its side; say, $\Delta_j = CD_1E_1$ is a triangle of the dissection (see Fig. 3). By our assumptions, Δ_j is not split. Thus, if CD_1E_1 has no neighbor of the same size, then some triangle overhangs at each of its vertices. We denote by $\Delta_{1'}$ the triangle that overhangs at C. The presence of $\Delta_{1'}$ implies that CD_2 is the side of some triangle CD_2F_1, i.e. E_1F_1 is a side of $\Delta_{1'}$. Repeating this argument for the triangles incident to A and B, we obtain that the three vertices of Δ_1 are internal points on the peripheries of three triangles $\Delta_{1'}$, $\Delta_{2'}$, $\Delta_{3'}$ as exhibited in Fig. 3.

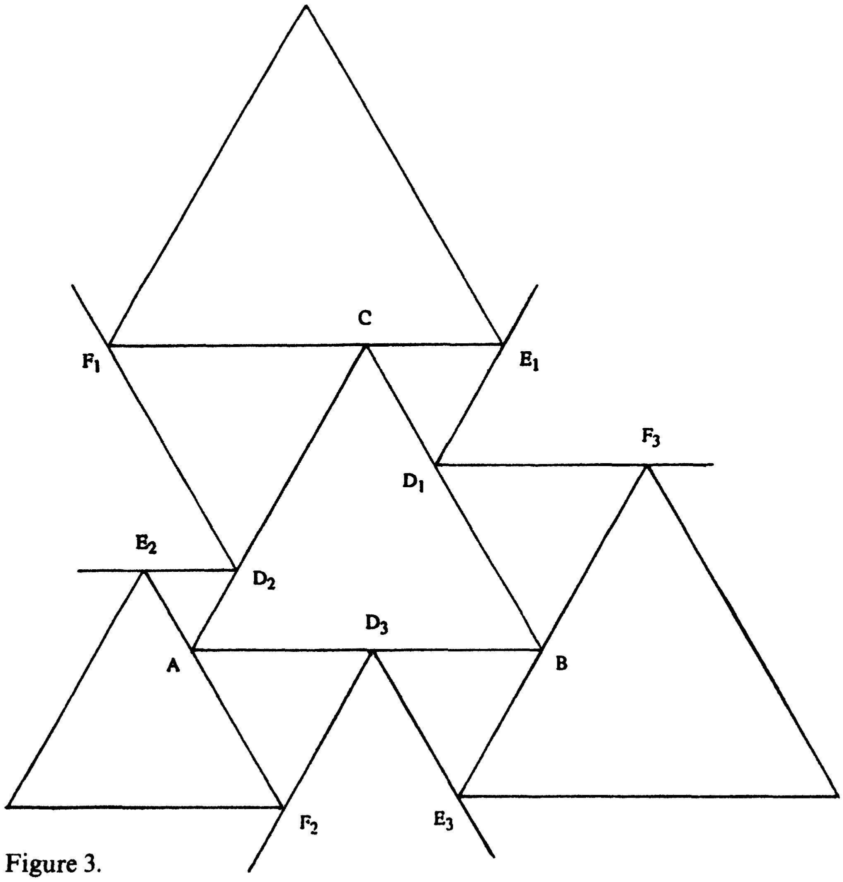

Figure 3.

Observing the six neighbors of Δ_1, we obtain $|CD_1| = |CE_1|$, $|CD_2| = |CF_1|$, etc., implying $|E_1F_1| + |E_2F_2| + |E_3F_3| = |AB| + |BC| + |CA|$ and $|\Delta_{1'}| + |\Delta_{2'}| + |\Delta_{3'}| = 3|\Delta_1|$. On the other hand, since Δ_1 is a smallest split triangle, $|\Delta_{i'}| \geq |\Delta_1|$ holds for $1 \leq i \leq 3$. Consequently, $|\Delta_{i'}| = |\Delta_1|$ for $1 \leq i \leq 3$, i.e. the dissection contains four mutually congruent triangles.

Case 4: There are at least two internal vertices on the same side of Δ_1.

Let D, E be two consecutive internal vertices in AB (see Fig. 2(c)). Since the triangle DEF is not split, and at most one of its neighbors can overhang at F, its other neighbor has the same size. This fact completes the proof of Theorem 1. □

Proof of Theorem 2.

We prove the theorem in the slightly different – but equivalent – form that the limit

$$\lim_{n \to \infty} (f(n) - 1)/(n - 1)$$

exists. Put $g(n) = (f(n) - 1)/(n - 1)$. We are going to show that

$$c := \liminf g(n) \geq g(k)$$

for all $k \geq 6$. This inequality will imply the theorem.

Suppose that two dissections D_k and D_n are available, where D_i $(i = k, n)$ has i tiles with $f(i)$ distinct sizes. Substituting D_k into the smallest triangle of D_n, we obtain a dissection D_{n+k-1} with $n + k - 1$ tiles and $f(n) + f(k) - 1$ or $f(n) + f(k)$ distinct sizes; the number of sizes after substitution is $f(n) + f(k)$ if and only if the smallest size in D_n occurs at least twice. Applying this substitution t times, we obtain:

(a) For every k, t, and n, $f(n + kt - t) \geq f(n) + tf(k) - t$, and
(b) if there is a dissection of Δ into k tiles of $f(k)$ distinct sizes such that the smallest size occurs at least twice, then $f(n + kt - t) \geq f(n) + tf(k) - 1$.

The reason is that the size of every triangle in the image of D_k is strictly smaller than the sizes in D_n.
Fixing the value of k $(k \geq 6)$, let $n = (t+1)(k-1) + i$, $0 \leq i \leq k-2$, with t tending to infinity. Then (a) implies

$$\begin{aligned} g(n) &= (f(n) - 1)/(n - 1) \\ &\geq (f(i + k - 1) - 1 + t(f(k) - 1))/(i + k - 1 + t(k - 1)) \\ &= (f(k) - 1)/(k - 1) - o(1) \\ &= g(k) - o(1). \end{aligned}$$

This inequality holds independently of the value of i, so that $\liminf g(n) \geq g(k)$ follows, implying $f(n) = cn + o(n)$ for some constant c.
The lower bound of $c \geq 5/7$ is obtained by repeatedly substituting the dissection D_{15} – taken from [4] – with 10 distinct sizes (the smallest one occurring twice), shown in Fig. 4. □

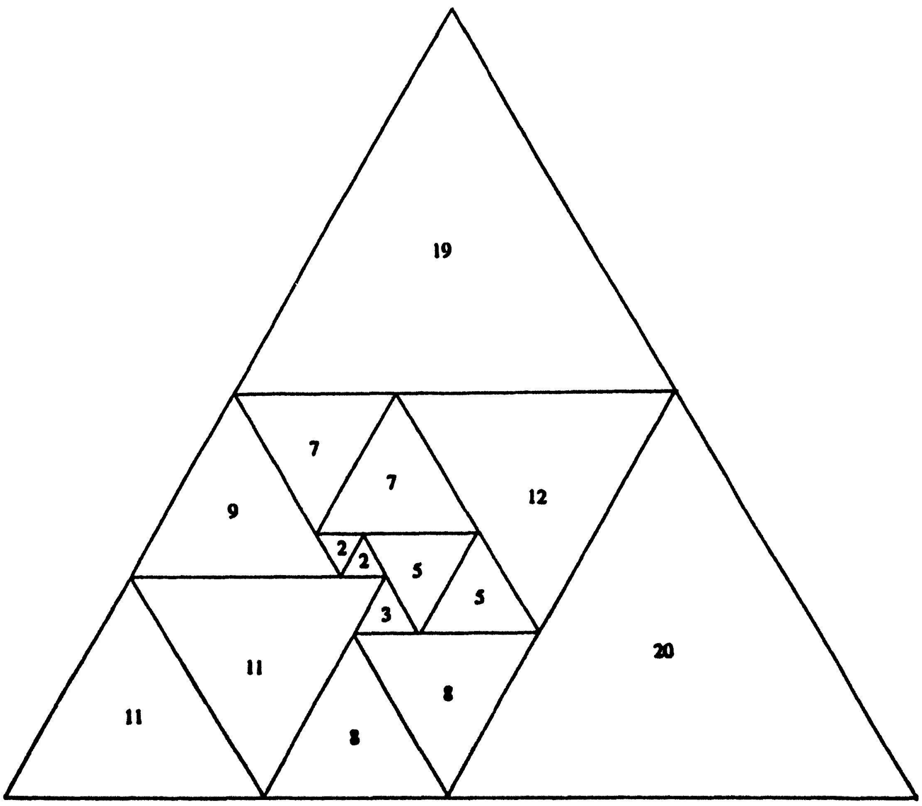

Figure 4. The dissection D_{15}.

As we have seen, property (a) is strong enough to prove the asymptotic result of Theorem 2. The advantage of part (b) is that it can be applied to obtain sharper lower bounds when a fairly good initial construction – like D_{15} – is available.

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Klassische Beleuchtungsgeometrie im E^d $(d \geqq 2)$
II. Kinematik in der Beleuchtungsgeometrie des E^d $(d \geqq 2)$

In Teil I (vgl. [6]) wurden im $\boldsymbol{R}^d$ $(d \geqq 2)$ Scharen aus isophotischen Flächenelementen bezüglich einer klassischen Zentralbeleuchtung $(\boldsymbol{q}, 1)$ – d.i. Lichtstärke 1 in jeder von der in $\boldsymbol{q}$ placierten Lichtquelle ausgehenden Richtung – konstruiert. Gemäss einer in [5] «kinematisch» genannten Vorgehensweise lassen sich nun aus der Zentralbeleuchtung $(\boldsymbol{q}, 1)$ und solch einer Schar isophotischer Elemente neue isophotische Scharen erzeugen. Diese Methode wird hier, nach kurzer Darlegung, verwendet, um Zusammenhänge zwischen Kurvenklassen aufzudecken, die die klassische Beleuchtungsgeometrie im E^d $(d \geqq 2)$ aufgrund ihres dimensionsabhängigen Beleuchtungsstärkegesetzes liefert.

1. Ein kinematisches Erzeugungsprinzip

Hat das orientierte Flächenelement $(\boldsymbol{x}, \boldsymbol{n})$ mit Trägerpunkt $\boldsymbol{x}$ und Einheitsnormalenvektor $\boldsymbol{n}$ die Beleuchtungsstärke $E > 0$ bezüglich der Zentralbeleuchtung $(\boldsymbol{q}, 1)$, so auch das Element $(\boldsymbol{q}, -\boldsymbol{n})$ bezüglich der Beleuchtung $(\boldsymbol{x}, 1)$. Die Ersetzung

$$((\boldsymbol{q}, 1), (\boldsymbol{x}, \boldsymbol{n})) \rightarrow ((\boldsymbol{x}, 1), (\boldsymbol{q}, -\boldsymbol{n}))$$