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LITERATUR

[1] Erdös P.: Problems and results in combinatorial geometry. Discrete geometry and convexity, Proc. Conf., New York 1982, Ann. N.Y. Acad. Sci. *440* (1985), 1–11.
[2] Giering O.: Analytische Behandlung des räumlichen Trilaterationsproblems $\{4, 6, 0, 0\}$. Deutsche Geod. Komm. Bayer. Akad. Wiss. *A/104* (1986), 3–38.
[3] Hopf H., Pannwitz E.: Jahresbericht d. Deutschen Mathem.-Vereinigung *43* (1934), 114, Aufgabe 167.
[4] Killian K., Meissl P.: Einige Grundaufgaben der räumlichen Trilateration und ihre gefährlichen Örter. Deutsche Geod. Komm. Bayer. Akad. Wiss. *A/61* (1969), 65–72.
[5] Palásti I.: On the seven points problem of P. Erdös. Stud. Sci. Math. Hung. *22* (1987), 447–448.
[6] Sutherland J. W.: Jahresbericht d. Deutschen Mathem.-Vereinigung *45* (1935), 33–35, Lösung d. Aufgabe 167.
[7] Wunderlich W.: Gefährliche Annahmen der Trilateration und bewegliche Fachwerke. I, II. ZAMM *57* (1977), 297–304 und 363–368.

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An extension of Blaschke's theorem in the plane

1. Introduction

Let K be a convex body in Euclidean d-space, E^d, having width $w(K) = w$ and inradius $r(K) = r$. The following theorem is well known (see for example pages 112–114 of Eggleston [1]):

Lemma 1. *(Blaschke's theorem.)*

$$w/r \le \begin{cases} 2\sqrt{d} & \text{for } d \text{ odd,} \\ 2(d+1)/\sqrt{d+2} & \text{for } d \text{ even,} \end{cases}$$

with equality when and only when K is a regular simplex.

Although the statement is relatively simple, Blaschke's theorem is difficult to prove, as the proof must take into account the different behaviour for even and odd dimension. Here we establish an analogue for Blaschke's theorem in the plane. A similar analogue may well exist for higher dimensions, but noting the difficulty of proving Blaschke's theorem, it is unclear how one might proceed.

Let K be a convex domain in the plane. We assert that K is contained in a trapezium T (perhaps degenerating to a triangle) which has the same inradius as K. For consider an incircle C of K. It is well-known that the boundary of K meets C either in diametrically opposite points, or in three points whose convex hull contains the centre of C in its interior. In either case, choose support lines to K at these points of contact: such lines will help determine T. In the first case we may choose any other pair of parallel support lines to K; with the given lines these will form a suitable trapezium (parallelogram) T. In the second case we have three lines determining the sides of an acute angled triangle. We choose a fourth support line to K, parallel to one of the triangle sides, and separated from it by K. These four lines now determine our trapezium T, possibly degenerating to a triangle.

Let T have successive vertices $A, A', B', B, (A)$, with $AA' // BB'$, and set $\|AA'\| = a$, $\|BB'\| = b$. Also set $b/a = t$ $(0 \leq t \leq 1)$. We prove:

Theorem. *For each* $t\,(0 \leq t \leq 1)$,

$$w/r + t \leq 3,$$

and this inequality cannot be improved.

We observe that for $t = 0$ the trapezium degenerates to a triangle, giving Blaschke's result $w/r \leq 3$ as a special case. In Blaschke's theorem, equality occurs precisely when K is a triangle (simplex). However, it is easy to see that K need not be a trapezium for equality in the above theorem.

The following lemma will be useful:

Lemma 2. *Amongst all triangles Δ with maximal side-length c and width w, the maximal inradius is attained (only) by the isosceles triangle with base of length c, and the minimal inradius is attained (only) by the isosceles triangle Δ^* with two equal sides of length c.*

Proof. Let ΔCDE be a general member of the family of triangles satisfying the given constraints and having c, d, e, $(c \geq d \geq e)$ as the opposite side-lengths. Then expressing the double area of the triangle in two ways, we have $(c + d + e)\,r = c\,w$. This shows that for fixed c and w, r is maximal (minimal) exactly when $d + e$ is minimal (maximal) under the restriction $c \geq d \geq e$.

Placing the triangle so that the side DE of length c lies on the x-axis, and vertex C on the line $y = w$, we see that the vertex C must also lie on the ellipse with foci D and E, and major axis of length $d + e$ along the x-axis. We thus obtain a family of confocal ellipses. Within this family, $d + e$ will be minimal for the ellipse which just touches the line $y = w$ – that is, for the isosceles triangle with base of length c. On the other hand, the vertex C will lie on the outermost permissible ellipse of the family when it lies on the line $y = w$ at its most asymmetric position relative to DE, under the constraints $c \geq d \geq e$. This occurs when $d = c$, and for this position $d + e$ will be maximal – that is, $d + e$ is maximal for either of the two congruent isosceles triangles of the family having two equal sides of length c. This completes the proof.

For later reference we notice that for the triangles Δ, Δ^* described in the lemma,

$$w(\Delta) = w(\Delta^*), \quad r(\Delta) \geq r(\Delta^*), \tag{1}$$

with equality in the second case only when $\Delta = \Delta^*$.

2. Some Preliminary Results

Since $K \subset T$, we have $w(K) \leq w(T)$; also we are given that $r(K) = r(T)$. Hence it is sufficient to establish the theorem for the trapezium T. We say that T is maximal if the expression $w(T)/r(T) + t(T)$ is maximal. The existence of such a maximal set is guaranteed by an easy application of the Blaschke selection theorem.

We may suppose that $t > 0$, (so T is not a triangle), else we have our result by Blaschke's theorem, or directly as follows:
Let T be a triangle with side lengths $c \geq d \geq e$, corresponding altitudes h_c, h_d, h_e, and inradius r. Then the width of T is given by $w = h_c$. As in Lemma 2, we obtain

$$(c + d + e)r = \tfrac{1}{3}(c h_c + d h_d + e h_e) \geq \tfrac{1}{3}(c + d + e)w,$$

whence $w \leq 3r$. Equality obviously holds if and only if $h_c = h_d = h_e$; $c = d = e$.
Assume for now that $0 \leq t \leq 1$.
Let edges AB, $A'B'$ of the trapezium T meet in V; then T is a subset of $\Delta = \Delta VAA'$. We may name the trapezium so that $\angle A' \geqq \angle A$. Let $w_p(T)$ denote the distance between the parallel edges of T. We notice that (for $t < 1$) an incircle of a maximal trapezium T cannot touch the two parallel edges. For then, $w(T) = w_p(T) = 2r(T)$, so $\omega/r + t = 2 + t < 3$, and T is not maximal. It follows that for maximal T,

$$r(T) < \tfrac{1}{2}\omega_p(T). \tag{2}$$

Further, since $r(T) = \min\{r(\Delta), \frac{1}{2}w_p(T)\}$, we deduce that for maximal T,

$$r(T) = r(\Delta). \tag{3}$$

Lemma 3. *If T is maximal, then $w(T) = w_p(T) = w(\Delta)$.*

Proof. Clearly $\omega(T) = \min\{w(\Delta), w_p(T)\}$. Suppose first that $w(\Delta) < w_p(T)$. We can then move B along BA a little way towards A, and B' a little way along $B'A'$ towards A', keeping $BB'//AA'$, and leaving both w and r unaltered. The length b, and so the number t, increases, thus increasing the expression $w/r + t$; hence T is not maximal.
If $w(\Delta) > w_p(T)$, we can rotate $VB'A'$ through a small angle about V, so that B', A' move along the lines BB', AA' towards B, A respectively. This preserves the ratio $a/b = t$, leaves $w(T)$ unchanged, and decreases $r(T)(= r(\Delta)$, by equation (3)). It follows that T is not maximal in this case either.
We deduce that for maximal T we must have $w(T) = w_p(T) = w(\Delta)$.

Corollary 3.1. *If T is maximal, and $t > 0$, then in triangle Δ, $\angle V < \angle A'$.*

Proof. The width of a triangle is the length of the shortest altitude, that is, the length of the altitude from the vertex with greatest angle. Since $w_p(T) = w(\Delta)$, such an altitude cannot have V as endpoint.

Lemma 4. *If T is maximal, then it is symmetric about a line perpendicular to its parallel edges.*

Proof. Let T be a given maximal trapezium inscribed in triangle Δ, and suppose that T is not symmetric. From our assumptions on the angles of T, and by Corollary 3.1, we have $\angle A < \angle A'$, $\angle V < \angle A'$. Using Lemma 2, we can replace $\Delta = \Delta VAA'$ by a new triangle $\Delta^* = \Delta VAY$, having $\angle V \leq \angle A = \angle Y$, and with Δ and Δ^* satisfying (1). Let Z be the point on VY such that $BZ//AY$, and let T^* denote the trapezium $AYZB(A)$. We compare the three numbers t, w and r for T^* and T.

(a) The ratio t. By similar triangles, $BB'/AA' (= t) = BZ/AY$, so $t(T) = t(T^*)$.
(b) The width w. Since $\pi/2 > \angle VBZ > \angle VBB'$,

$$w_p(T) < w_p(T^*). \tag{4}$$

By (4), Lemma 3, and (1),

$$w_p(T^*) > w_p(T) = w(T) = w(\Delta) = w(\Delta^*). \tag{5}$$

Hence

$$w(T^*) = \min\{w_p(T^*), w(\Delta^*)\} = w(T).$$

(c) The inradius r. From (1), (3), (2) and (4),

$$r(\Delta^*) < r(\Delta) = r(T) < \tfrac{1}{2}\omega_p(T) < \tfrac{1}{2}w_p(T^*).$$

Hence

$$r(T^*) = \min\{r(\Delta^*), \tfrac{1}{2}w_p(T^*)\} < r(T).$$

From comparing these three quantities, we deduce that the given trapezium T cannot be maximal. Hence if T is maximal, it must be a symmetric trapezium.

3. Proof of the Theorem

It follows that we may henceforth assume that trapezium T with vertices A, A', B', B is symmetric about a line perpendicular to the parallel edges, and that $w = w(T) = w_p(T) = w(\Delta)$.

Now set $\theta = \angle VAA'$, and let h denote the altitude of Δ from vertex V. Then by similarity and simple trigonometry

$$a/h = b/(h-w), \quad w = a\sin\theta, \quad h = (a/2)\tan\theta, \quad r = (a/2)\tan(\theta/2).$$

Substituting for w and h in the first of these expressions give

$$b = a(1 - 2\cos\theta); \tag{6}$$

also

$$w/r = 4\cos^2(\theta/2).$$

Eliminating θ between these two expressions gives

$$w/r + b/a = w/r + t = 3.$$

For any t $(0 < t < 1)$ we can construct a symmetric trapezium T satisfying the conditions of Lemma 3, and having $b/a = t$: simply choose θ to satisfy equation (6). We deduce that for each value of $t\,(0 \leq t < 1)$, the inequality $w/r + t \leq 3$ holds, with equality for the corresponding maximal trapezium T.
Finally, we observe that in the limit as $t \to 1$, the symmetric trapezium T assumes the form of a square of side length w. In this case the theorem gives $w/r = 2$ as expected.
I am grateful to the referee for some valuable suggestion on the presentation of this paper.

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REFERENCES

1 Eggleston H. G., Convexity, Cambridge Tract No. 47, Cambridge University Press (1963).

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Perfekte Dreieckzerlegungen

Vor 50 Jahren konstruierte R. Sprague [6] zum ersten Mal eine Zerlegung eines Quadrates in paarweise inkongruente Quadrate (55 Teile). Solche Zerlegungen wurden in der Folgezeit *perfekt* genannt, und eine Reihe von Autoren (vgl. die umfassende Literaturübersicht in [4]) suchte nach perfekten Quadratzerlegungen in möglichst wenige Teile. 1978 fand A. J. W. Duijvestijn [3] eine perfekte Zerlegung eines Quadrates in 21 Teile, und mit Hilfe von Computerprogrammen konnte er beweisen, dass damit die minimale Teilezahl erreicht ist. Bereits in [2] wurde bemerkt, dass sich gleichseitige Dreiecke bzw. Würfel nicht in paarweise inkongruente gleichseitige Dreiecke (ohne Beweis) bzw. Würfel (mit Beweis) zerlegen lassen. Naheliegend ist die folgende

Definition. Eine elementar-geometrische Zerlegung eines d-Polyeders Π des d-dimensionalen euklidischen Raumes in Polyeder $\Pi_1, \Pi_2, \ldots, \Pi_i$ $(d \geq 2, i \geq 2)$ heisst genau dann *perfekt*, wenn sie die folgenden beiden Eigenschaften hat:

(i) Π_j ist ähnlich zu Π für $j \in \{1, 2, \ldots, i\}$.
(ii) Π_j und Π_k sind inkongruent für $j, k \in \{1, 2, \ldots, i\}$ und $j \neq k$.

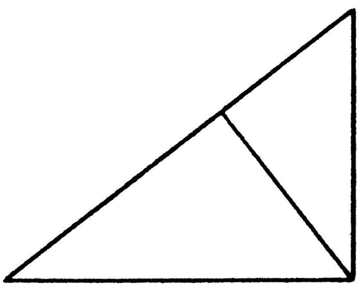
Fig. 1

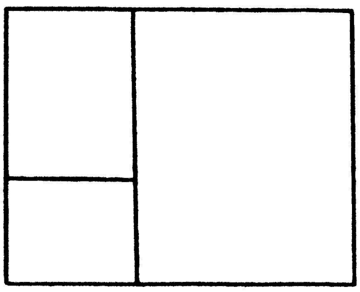
Fig. 2