Kleine Mitteilungen

On an integral inequality

1. Introduction

The following is an interesting elementary inequality which was posed as a problem of the 34th Putnam Competition 1973:
If f is differentiable on $[0,1]$ with $f(0)=0$ and $0<f'(x)\le 1$ for all $x\in[0,1]$ then

$$\int_0^1 f^3(x)\,dx \le \left(\int_0^1 f(x)\,dx\right)^2. \tag{1}$$

Furthermore, the following inequality appeared as Problem P338 in Canad. Math. Bull. *26* (June 1983):
Let $f:[0,1]\to \boldsymbol{R}$ be differentiable such that $f(0)=0$ and $0\le f'(x)\le 1$ whenever $x\in[0,1]$. If $p\ge 1$ then

$$\left(\int_0^1 f(x)\,dx\right)^p \ge p\,2^{1-p}\int_0^1 f(x)^{2p-1}\,dx. \tag{2}$$

For $0<p<1$ the reverse inequality is valid.
Of course for $p=2$, (2) becomes (1).
In this note we shall prove extensions of inequalities (1) and (2).

2. Results

i) Let $a>0$ and $w:[0,a]\to[0,\infty)$ be an integrable function and $f:[0,a]\to[0,\infty)$ be differentiable such that $f(0)=0$. We define

$$F(x):=\left(\int_0^x w(t)\,f(t)\,dt\right)^p - p\,2^{1-p}\int_0^x w(t)\,f(t)^{2p-1}\,dt. \tag{3}$$

Then $F(0)=0$ and

$$F'(x)=p\,w(x)\,f(x)\left[\left(\int_0^x w(t)\,f(t)\,dt\right)^{p-1} - 2^{1-p}f(x)^{2(p-1)}\right].$$

Let $p>1$ and put

$$G(x):=\int_0^x w(t)\,f(t)\,dt - f^2(x)/2.$$

Then $G(0)=0$ and $G'(x)=f(x)[w(x)-f'(x)]$. If there holds

$$0 \le f'(x) \le w(x) \tag{4}$$

we get $G'(x) \ge 0$. Thus we conclude $G(x) \ge 0$, $F'(x) \ge 0$ and finally $F(x) \ge 0$. If

$$f'(x) \ge w(x) \tag{5}$$

we get $G'(x) \le 0$ and as above $F(x) \le 0$.
Analogously we can consider the cases $0<p<1$ or $p<0$. Therefore the following theorem is valid.

Theorem 1

Let f: $[0,a] \to \boldsymbol{R}$ be differentiable such that $f(0)=0$. Then

$$\left(\int_0^a w(x) f(x)\,dx\right)^p \ge p\,2^{1-p} \int_0^a w(x) f(x)^{2p-1}\,dx \tag{6}$$

if $0 \le f'(x) \le w(x)$ and $p>1$ or $p<0$ or $f'(x) \ge w(x)$ and $0<p<1$. The reverse inequality is valid for $0 \le f'(x) \le w(x)$ and $0<p<1$ or $f'(x) \ge w(x)$ and $p>1$ or $p<0$. ∎

($a=1$ and $w(x) \equiv 1$ yield inequality (2).)

ii) For another generalisation of (1) let $M>0$ and

$$f(0)=0 \quad \text{and} \quad 0 \le f'(x) \le M \quad \text{for all} \quad x \in [0,a]. \tag{7}$$

Then $0 \le f(x) \le Mx$ and $0 \le \int_0^x f(t)\,dt \le M x^2/2$ for $0 \le x \le a$.
We now define (for suitable p and r)

$$F(x) := \left(\int_0^x f(t)\,dt\right)^p - \int_0^x f(t)^r\,dt.$$

Then $F(0)=0$ and $F'(x)=f(x)g(x)$, where

$$g(x) = p\left(\int_0^x f(t)\,dt\right)^{p-1} - f(x)^{r-1}.$$

Clearly, $g(0)=0$ and

$$g'(x) = f(x)\left[p(p-1)\left(\int_0^x f(t)\,dt\right)^{p-2} - (r-1) f(x)^{r-3} f'(x)\right].$$

Let $1 < p \leq 2$ and $r \geq 3$. Then

$$g'(x) \geq f(x)[p(p-1)(Mx^2/2)^{p-2} - (r-1)M^{r-2}x^{r-3}]$$
$$= f(x)x^{2p-4}M^{r-2}[p(p-1)2^{2-p}M^{p-r} - (r-1)x^{r-2p+1}].$$

Thus, if

$$0 < a \leq [p(p-1)2^{2-p}M^{p-r}/(r-1)]^{1/(r-2p+1)} \tag{8}$$

we have $g'(x) \geq 0$, $g(x) \geq 0$, $F'(x) \geq 0$ and finally $F(x) \geq 0$ Therefore we have proved the following

Theorem 2

Let $1 < p \leq 2$ *and* $r \geq 3$. *The differentiable function* $f\colon [0,a] \to \mathbf{R}$ *satisfies* $f(0) = 0$ *and* $0 \leq f'(x) \leq M$ *for all* $0 \leq x \leq a$, *a subject to* (8). *Then*

$$\left(\int_0^a f(x)\,dx\right)^p \geq \int_0^a f(x)^r\,dx. \tag{9}$$

If $f'(x) \geq M$ *the reverse inequality holds true.* ■

For $M = 1$ we have a result of P. R. Beesack (see [1]).

Remark. For another generalisation of (1) see [2] and [3].

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