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# Kleine Mitteilungen

## On an integral inequality

### 1. Introduction

The following is an interesting elementary inequality which was posed as a problem of the 34th Putnam Competition 1973:

If  $f$  is differentiable on  $[0, 1]$  with  $f(0) = 0$  and  $0 < f'(x) \leq 1$  for all  $x \in [0, 1]$  then

$$\int_0^1 f^3(x) dx \leq \left( \int_0^1 f(x) dx \right)^2. \tag{1}$$

Furthermore, the following inequality appeared as Problem P338 in *Canad. Math. Bull.* 26 (June 1983):

Let  $f: [0, 1] \rightarrow \mathbf{R}$  be differentiable such that  $f(0) = 0$  and  $0 \leq f'(x) \leq 1$  whenever  $x \in [0, 1]$ . If  $p \geq 1$  then

$$\left( \int_0^1 f(x) dx \right)^p \geq p 2^{1-p} \int_0^1 f(x)^{2p-1} dx. \tag{2}$$

For  $0 < p < 1$  the reverse inequality is valid.

Of course for  $p = 2$ , (2) becomes (1).

In this note we shall prove extensions of inequalities (1) and (2).

### 2. Results

i) Let  $a > 0$  and  $w: [0, a] \rightarrow [0, \infty)$  be an integrable function and  $f: [0, a] \rightarrow [0, \infty)$  be differentiable such that  $f(0) = 0$ . We define

$$F(x) := \left( \int_0^x w(t) f(t) dt \right)^p - p 2^{1-p} \int_0^x w(t) f(t)^{2p-1} dt. \tag{3}$$

Then  $F(0) = 0$  and

$$F'(x) = p w(x) f(x) \left[ \left( \int_0^x w(t) f(t) dt \right)^{p-1} - 2^{1-p} f(x)^{2(p-1)} \right].$$

Let  $p > 1$  and put

$$G(x) := \int_0^x w(t) f(t) dt - f^2(x)/2.$$

Then  $G(0) = 0$  and  $G'(x) = f(x)[w(x) - f'(x)]$ . If there holds

$$0 \leq f'(x) \leq w(x) \quad (4)$$

we get  $G'(x) \geq 0$ . Thus we conclude  $G(x) \geq 0$ ,  $F'(x) \geq 0$  and finally  $F(x) \geq 0$ . If

$$f'(x) \geq w(x) \quad (5)$$

we get  $G'(x) \leq 0$  and as above  $F(x) \leq 0$ .

Analogously we can consider the cases  $0 < p < 1$  or  $p < 0$ . Therefore the following theorem is valid.

### Theorem 1

Let  $f: [0, a] \rightarrow \mathbf{R}$  be differentiable such that  $f(0) = 0$ . Then

$$\left( \int_0^a w(x) f(x) dx \right)^p \geq p 2^{1-p} \int_0^a w(x) f(x)^{2p-1} dx \quad (6)$$

if  $0 \leq f'(x) \leq w(x)$  and  $p > 1$  or  $p < 0$  or  $f'(x) \geq w(x)$  and  $0 < p < 1$ . The reverse inequality is valid for  $0 \leq f'(x) \leq w(x)$  and  $0 < p < 1$  or  $f'(x) \geq w(x)$  and  $p > 1$  or  $p < 0$ . ■

( $a = 1$  and  $w(x) \equiv 1$  yield inequality (2).)

ii) For another generalisation of (1) let  $M > 0$  and

$$f(0) = 0 \quad \text{and} \quad 0 \leq f'(x) \leq M \quad \text{for all} \quad x \in [0, a]. \quad (7)$$

Then  $0 \leq f(x) \leq Mx$  and  $0 \leq \int_0^x f(t) dt \leq Mx^2/2$  for  $0 \leq x \leq a$ .

We now define (for suitable  $p$  and  $r$ )

$$F(x) := \left( \int_0^x f(t) dt \right)^p - \int_0^x f(t)^r dt.$$

Then  $F(0) = 0$  and  $F'(x) = f(x)g(x)$ , where

$$g(x) = p \left( \int_0^x f(t) dt \right)^{p-1} - f(x)^{r-1}.$$

Clearly,  $g(0) = 0$  and

$$g'(x) = f(x) \left[ p(p-1) \left( \int_0^x f(t) dt \right)^{p-2} - (r-1) f(x)^{r-3} f'(x) \right].$$

Let  $1 < p \leq 2$  and  $r \geq 3$ . Then

$$\begin{aligned} g'(x) &\geq f(x)[p(p-1)(Mx^2/2)^{p-2} - (r-1)M^{r-2}x^{r-3}] \\ &= f(x)x^{2p-4}M^{r-2}[p(p-1)2^{2-p}M^{p-r} - (r-1)x^{r-2p+1}]. \end{aligned}$$

Thus, if

$$0 < a \leq [p(p-1)2^{2-p}M^{p-r}/(r-1)]^{1/(r-2p+1)} \tag{8}$$

we have  $g'(x) \geq 0$ ,  $g(x) \geq 0$ ,  $F'(x) \geq 0$  and finally  $F(x) \geq 0$  Therefore we have proved the following

**Theorem 2**

Let  $1 < p \leq 2$  and  $r \geq 3$ . The differentiable function  $f: [0, a] \rightarrow \mathbf{R}$  satisfies  $f(0) = 0$  and  $0 \leq f'(x) \leq M$  for all  $0 \leq x \leq a$ , a subject to (8). Then

$$\left(\int_0^a f(x) dx\right)^p \geq \int_0^a f(x)^r dx. \tag{9}$$

If  $f'(x) \geq M$  the reverse inequality holds true. ■

For  $M = 1$  we have a result of P. R. Beesack (see [1]).

**Remark.** For another generalisation of (1) see [2] and [3].

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