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Kleine Mitteilungen

On an integral inequality

1. Introduction

The following is an interesting elementary inequality which was posed as a problem of the 34th Putnam Competition 1973:

If f is differentiable on $[0, 1]$ with $f(0) = 0$ and $0 < f'(x) \leq 1$ for all $x \in [0, 1]$ then

$$\int_0^1 f^3(x) dx \leq \left(\int_0^1 f(x) dx \right)^2. \quad (1)$$

Furthermore, the following inequality appeared as Problem P338 in Canad. Math. Bull. 26 (June 1983):

Let $f: [0, 1] \rightarrow \mathbf{R}$ be differentiable such that $f(0) = 0$ and $0 \leq f'(x) \leq 1$ whenever $x \in [0, 1]$. If $p \geq 1$ then

$$\left(\int_0^1 f(x) dx \right)^p \geq p 2^{1-p} \int_0^1 f(x)^{2p-1} dx. \quad (2)$$

For $0 < p < 1$ the reverse inequality is valid.

Of course for $p = 2$, (2) becomes (1).

In this note we shall prove extensions of inequalities (1) and (2).

2. Results

i) Let $a > 0$ and $w: [0, a] \rightarrow [0, \infty)$ be an integrable function and $f: [0, a] \rightarrow [0, \infty)$ be differentiable such that $f(0) = 0$. We define

$$F(x) := \left(\int_0^x w(t) f(t) dt \right)^p - p 2^{1-p} \int_0^x w(t) f(t)^{2p-1} dt. \quad (3)$$

Then $F(0) = 0$ and

$$F'(x) = p w(x) f(x) \left[\left(\int_0^x w(t) f(t) dt \right)^{p-1} - 2^{1-p} f(x)^{2(p-1)} \right].$$

Let $p > 1$ and put

$$G(x) := \int_0^x w(t) f(t) dt - f^2(x)/2.$$

Then $G(0) = 0$ and $G'(x) = f(x)[w(x) - f'(x)]$. If there holds

$$0 \leq f'(x) \leq w(x) \quad (4)$$

we get $G'(x) \geq 0$. Thus we conclude $G(x) \geq 0$, $F'(x) \geq 0$ and finally $F(x) \geq 0$. If

$$f'(x) \geq w(x) \quad (5)$$

we get $G'(x) \leq 0$ and as above $F(x) \leq 0$.

Analogously we can consider the cases $0 < p < 1$ or $p < 0$. Therefore the following theorem is valid.

Theorem 1

Let $f: [0, a] \rightarrow \mathbf{R}$ be differentiable such that $f(0) = 0$. Then

$$\left(\int_0^a w(x) f(x) dx \right)^p \geq p 2^{1-p} \int_0^a w(x) f(x)^{2p-1} dx \quad (6)$$

if $0 \leq f'(x) \leq w(x)$ and $p > 1$ or $p < 0$ or $f'(x) \geq w(x)$ and $0 < p < 1$. The reverse inequality is valid for $0 \leq f'(x) \leq w(x)$ and $0 < p < 1$ or $f'(x) \geq w(x)$ and $p > 1$ or $p < 0$. ■

($a = 1$ and $w(x) \equiv 1$ yield inequality (2).)

ii) For another generalisation of (1) let $M > 0$ and

$$f(0) = 0 \quad \text{and} \quad 0 \leq f'(x) \leq M \quad \text{for all } x \in [0, a]. \quad (7)$$

Then $0 \leq f(x) \leq Mx$ and $0 \leq \int_0^x f(t) dt \leq Mx^2/2$ for $0 \leq x \leq a$.

We now define (for suitable p and r)

$$F(x) := \left(\int_0^x f(t) dt \right)^p - \int_0^x f(t)^r dt.$$

Then $F(0) = 0$ and $F'(x) = f(x)g(x)$, where

$$g(x) = p \left(\int_0^x f(t) dt \right)^{p-1} - f(x)^{r-1}.$$

Clearly, $g(0) = 0$ and

$$g'(x) = f(x) \left[p(p-1) \left(\int_0^x f(t) dt \right)^{p-2} - (r-1) f(x)^{r-3} f'(x) \right].$$

Let $1 < p \leq 2$ and $r \geq 3$. Then

$$\begin{aligned} g'(x) &\geq f(x)[p(p-1)(Mx^2/2)^{p-2} - (r-1)M^{r-2}x^{r-3}] \\ &= f(x)x^{2p-4}M^{r-2}[p(p-1)2^{2-p}M^{p-r} - (r-1)x^{r-2p+1}]. \end{aligned}$$

Thus, if

$$0 < a \leq [p(p-1)2^{2-p}M^{p-r}/(r-1)]^{1/(r-2p+1)} \quad (8)$$

we have $g'(x) \geq 0$, $g(x) \geq 0$, $F'(x) \geq 0$ and finally $F(x) \geq 0$. Therefore we have proved the following

Theorem 2

Let $1 < p \leq 2$ and $r \geq 3$. The differentiable function $f: [0, a] \rightarrow \mathbb{R}$ satisfies $f(0) = 0$ and $0 \leq f'(x) \leq M$ for all $0 \leq x \leq a$, a subject to (8). Then

$$\left(\int_0^a f(x) dx\right)^p \geq \int_0^a f(x)^r dx. \quad (9)$$

If $f'(x) \geq M$ the reverse inequality holds true. ■

For $M = 1$ we have a result of P. R. Beesack (see [1]).

Remark. For another generalisation of (1) see [2] and [3].

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