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wird und $2m/(1+u)-(m-1)$ beträgt. Weiter ist $\sigma'(\alpha_0) = 2mf'(\alpha_0) - (m-1) = 0$, d. h. $f'(\alpha_0) = (m-1)/2m$. Auf $[0, \alpha_0)$ ist $\sigma \geq 0$ (weil $\sigma' > 0$) und $\sigma \leq \alpha$ (weil $f' \leq \frac{1}{2}$, somit $\sigma' \leq m - (m-1) = 1$). Da die Klammer von (3.17) monoton mit $\sigma$ und $\sigma$ monoton mir $\alpha$ wächst, wird ihr Maximum in $\alpha_0$ erreicht. Es beträgt, wegen $\sigma(\alpha_0) \leq \alpha_0$, höchstens $\frac{1}{2} - f'(\alpha_0) = 1/2m$. Insgesamt bleibt also der Ausdruck in (3.17) kleiner als

$$\left[\frac{2m}{1+u} - (m-1)\right] \cdot \frac{1}{2m} - \frac{1}{2m} = \frac{1-u}{2(1+u)} < 0.$$

Damit ist der Beweis beendet. Dieses letzte Beispiel lehrt, dass man starke Symmetrieeigenschaften des Graphen postulieren muss, um die Nullstellen an $\mathbb{R}_-$ oder mindestens $\mathbb{R}_- \cup S_1$ zu binden. Für weitere Diskussionen vgl. [2] und [3].

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LITERATURVERZEICHNIS

1 Onsager L.: Crystal Statistics I. A Two-Dimensional Model with an Order-Disorder Transition. Phys. Rev. *65*, 117 (1944).
2 Peitgen H.-O., Richter P. H.: The Beauty of Fractals. Springer Berlin, Heidelberg, New York, Tokyo 1986.
3 V. Saarloos W., Kurtze D. A.: Location of zeros in the complex temperature plane. Absence of Lee-Yang theorem. J. Phys. *A 17*, 1301 (1984).
4 Yang C. N., Lee T. D.: Statistical Theory of Equations of State and Phase Transitions I, II. Phys. Rev. *87*, 404, 410 (1952).

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# Kleine Mitteilung

## On a problem by Shapiro

In 1954, H. S. Shapiro proposed the following problem: determine the minimum of the cyclic sum

$$S_n(a) = \sum_{i=1}^{n} a_i/(a_{i+1} + a_{i+2}), \tag{1}$$

with $a_i \geq 0$ and $a_i + a_{i+1} > 0$ ($a_{n+1} = a_1$, $a_{n+2} = a_2$). He conjectured that this minimum is $n/2$. This is known to be true for $n \leq 13$, but false for $n \geq 14$ (see [1] for reference, and the range $15 \leq n \leq 23$, $n$ odd).
In the present work we propose a method which enables us to solve the problem for $n < 7$. The method is based on an application of Fourier transformation to functions of discrete argument.

We denote

$$S = \frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_4} + \ldots + \frac{a_{n-1}}{a_n + a_1} + \frac{a_n}{a_1 + a_2}$$

$$\alpha_i = \frac{n a_i}{\sum_{k=1}^{n} a_k} - 1, \quad i = 1, 2, \ldots, n.$$

Then

$$\alpha_i > -1, \quad \sum_{k=1}^{n} \alpha_k = 0 \tag{2}$$

and

$$2S = \sum_{k=1}^{n} \frac{1 + \alpha_k}{1 + \dfrac{\alpha_{k+1} + \alpha_{k+2}}{2}}, \quad \alpha_{n+1} = \alpha_1, \ \alpha_{n+2} = \alpha_2 .$$

For $x > -1$, $(1 + x)^{-1} \geq 1 - x$, the equality being valid only for $x = 0$.
It follows

$$2S \geq \sum_{k=1}^{n} (1 + \alpha_k)\left(1 - \frac{\alpha_{k+1} + \alpha_{k+2}}{2}\right) = n - \frac{1}{2} \sum_{k=1}^{n} \alpha_k (\alpha_{k+1} + \alpha_{k+2}) \equiv n - \frac{A}{2}. \tag{3}$$

We introduce a real valued function $f(x)$, such that

$$f(k) = \alpha_k \quad \text{for} \quad 1 \leq k \leq n \quad \text{and} \quad f(k) = f(n + k).$$

From (2) it follows

$$\sum_{k=1}^{n} f(k) = 0. \tag{4}$$

We represent $f(x)$ by a Fourier trigonometric polynomial

$$f(k) = \sum_{m=1}^{n} q(m) \exp\left(2\pi i \frac{mk}{n}\right), \tag{5}$$

where

$$q(m) = \frac{1}{n} \sum_{k=1}^{n} f(k) \exp\left(-2\pi i \frac{mk}{n}\right) \tag{6}$$

Bearing in mind that $f(x)$ is real valued, we obtain

$$q(n - m) = \overline{q(m)}. \tag{7}$$

Besides, from (5) and (6) it follows

$$q(n+m) = q(m), \quad q(n) = 0. \tag{8}$$

Now, the sum $A$ can be represented as follows

$$A = n \sum_{k=1}^{n} q(n-k)\, q(k) \left( \cos \frac{2\pi}{n} k + \cos \frac{4\pi}{n} k \right)$$

$$+ i n \sum_{k=1}^{n} q(n-k)\, q(k) \left( \sin \frac{2\pi}{n} k + \sin \frac{4\pi}{n} k \right).$$

Because of (7) and (8), we have ($A$ being real)

$$A = n \sum_{k=1}^{n-1} q(n-k)\, q(k) \left( \cos \frac{2\pi}{n} k + \cos \frac{4\pi}{n} k \right).$$

Hence, by (3) and (7) it follows

$$S \geq \frac{n}{2} - \frac{n}{4} \sum_{k=1}^{n-1} |q(k)|^2 \left( \cos \frac{2\pi}{n} k + \cos \frac{4\pi}{n} k \right). \tag{9}$$

The function $\cos x + \cos 2x$ is non-positive for $\frac{\pi}{3} \leq x \leq \frac{5\pi}{3}$. Thus the second term of the last inequality (9) is non-negative for every $1 \leq k < n$ and $n < 7$.
Thus (1) is proved for $n < 7$.
The equality in (1) is obtained if and only if

$$\alpha_1 + \alpha_2 = \alpha_2 + \alpha_3 = \ldots = \alpha_{n-1} + \alpha_n = \alpha_n + \alpha_1 = 0.$$

For $n = 3, 5$ the last system has the following solution:

$$\alpha_k = 0, \quad k = 1, \ldots, n.$$

For $n = 4, 6$

$$\alpha_1 = \alpha_3 = \alpha_5; \quad \alpha_2 = \alpha_4 = \alpha_6 \quad \text{and hence}$$

$$\alpha_k = (-1)^k \alpha, \quad |\alpha| < 1.$$

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REFERENCES

[1] Troesch B. A.: The validity of Shapiro's cyclic inequality, Math. Comp. *53*, 657–664 (1989).


0013-6018/90/050137-03$1.50 + 0.20/0