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LITERATURVERZEICHNIS

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Kleine Mitteilung

Zeros of characters and the Frattini subgroup

Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of its (complex) irreducible characters. Of course, the Frattini subgroup $\Phi(G)$ being normal in G, it must be an intersection of certain kernels of elements of $\mathrm{Irr}(G)$. However, it seems that the problem of describing $\Phi(G)$ in terms of characters is still open. The aim of this short Note ist to give a sufficient condition for the nontriviality of $\Phi(G)$ in terms of vanishing sets of nonlinear irreducible characters of G.

Our notation is standard and follows that of [2]. Throughout, G will be a finite group and $Z(G)$, G' will denote its centre and its derived subgroup, respectively. If $\chi \in \mathrm{Irr}(G)$ is nonlinear, the vanishing set $A(\chi)$ of χ is defined by $A(\chi) := \{g \in G / \chi(g) = 0\}$. A well-known result of Burnside asserts that $A(\chi) \neq \Phi$; moreover, it's clear that $A(\chi)$ is a union of conjugate classes of elements of G.

We prove the following

Theorem. *Let G be a finite group with $1 < Z(G) < G$. Suppose that there exists a nonlinear $\chi \in Irr(G)$ such that $A(\chi)$ contains fewer than $|Z(G)|$ conjugate classes of elements. Then $\Phi(G) \neq 1$.*

Proof: The key observation is that actually $A(\chi)$ is a union of cosets modulo $Z(G)$. To prove this, note that by Problem 3.12 of [2] it follows that for every $g \in G$,

$$|\chi(g)|^2 = \frac{\chi(1)}{|G|} \sum_{n \in G} \chi([g, h]) . \qquad (*)$$

Let now $g, h \in G$ and $z \in Z(G)$; since $[g, h] = [gz, h]$, it follows from (*) that $g \in A(\chi)$ iff $gz \in A(\chi)$ for every $z \in Z(G)$. This means that $A(\chi)$ is a union of cosets modulo $Z(G)$.

Suppose, by way of contradiction, that $\Phi(G) = 1$. By a well-known result of [1], $G' \cap Z(G) \leq \Phi(G)$, so $G' \cap Z(G) = 1$.

Denote by $s(\chi)$ and $t(\chi)$ the number of conjugate classes of G contained in $A(\chi)$ and the number of cosets modulo $Z(G)$ lying in $A(\chi)$, respectively. We shall reach a contradiction by applying the pigeonhole principle. Suppose that $g, h \in A(\chi)$, $g \neq h$ and there exist $u \in G$ and $z \in Z(G)$ such that $g = h^u = hz$. Then $z = h^{-1} g = h^{-1} h^u = [h, u] \in G' \cap Z(G)$, whence $g = h$. This contradicts the choice of g and h and shows that $A(\chi)$ contains at most $s(\chi)\, t(\chi)$

elements. Taking now into account that $t(\chi) := |A(\chi)|/|Z(G)|$, it results that $|Z(G)| \geqq s(\chi)$. But this contradicts the hypothesis and we are done.

Corollary. *Let G be a finite group with $1 < Z(G) < G$ and $\Phi(G) = 1$. Let*

$$c := \max\{|C_G(x)|/x \in G \backslash Z(G)\} \quad \textit{and} \quad b(G) := \max\{\chi(1)/\chi \in \mathrm{Irr}(G)\}.$$

Then $b(G) \geq \left(\frac{c}{c - |Z(G)|}\right)^{\frac{1}{2}}$.

Proof: As a direct consequence of the Theorem, we obtain that $c|A(\chi)| \geq |Z(G)||G|$ for every nonlinear $\chi \in \mathrm{Irr}(G)$. On the other hand, it's a simple exercise to prove that $|A(\chi)| < |G| \frac{\chi(1)^2 - 1}{\chi(1)^2}$. Since $\frac{(b(G))^2 - 1}{(b(G))^2} \geq \frac{\chi(1)^2 - 1}{\chi(1)^2}$ for every $\chi \in \mathrm{Irr}(G)$, the result follows by combining these inequalities.

We have already seen that if $\chi \in \mathrm{Irr}(G)$ is nonlinear, then $|Z(G)| \,|\, |A(\chi)|$. This result may be refined in certain very special cases. For example, suppose that G is a finite group and $\chi \in \mathrm{Irr}(G)$ is faithful such that $\chi(1) = 2$ and $\chi(g) \in \mathbb{Q}$ for every $g \in G$. It is a matter of simple calculations to show that $|A(\chi)| = 3|Z(G)|$. If, moreover, $Z(G) = 1$, then $|A(\chi)| = 3$ and G has a maximal subgroup of index 3 (the centralizer of an element of $A(\chi)$).

Marian Deaconescu, Department of Mathematics,
University of Timisoara, Romania

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Aufgaben

Aufgabe 1009. n Zahlen $x_1, \ldots, x_n$ mit $x_i \in \{0, 1, \ldots, k\}$ $(k \geq 2)$ werden einmal linear, ein andermal kreisförmig so angeordnet, dass die Summe zweier Nachbarglieder stets von $k + 1$ verschieden ist. Für beide Fälle bestimme man die Anzahl der zulässigen Anordnungen.

J. Binz, Bolligen

Lösung des Aufgabenstellers (Bearbeitung der Redaktion).

s_n bzw. t_n bezeichnen die Anzahlen der zulässigen linearen bzw. kreisförmigen Anordnungen.

a) Es gilt $s_n = a_n + b_n$, wobei a_n die Anzahl der auf 0 endenden, b_n diejenige der übrigen zulässigen linearen Anordnungen bedeuten. Dann ist

$$a_{n+1} = a_n + b_n, \quad b_{n+1} = k a_n + (k-1) b_n, \quad a_2 = k + 1, \quad b_2 = k^2,$$