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Autor: Gunther, G.

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4.4. Sei $t < n$, $n \equiv 0 \pmod 2$ *und* $(a,n) \equiv 1 \pmod 2$. Wegen $n \equiv 0 \pmod 2$ und $(a,n) \equiv 1 \pmod 2$ hat man $a \equiv 1 \pmod 2$. Für ein $b \in \mathbb{Z}$ folgt aus $(b,n) = 1$ ebenso $b \equiv 1 \pmod 2$. Eine mögliche zulässige Darstellung $a \equiv b_1 + b_2 \pmod n$ von a würde also auf den Widerspruch $1 \equiv 1 + 1 \pmod 2$ führen. Wegen $\mu_n(a) \geqq 2$ (da $t < n$) ist somit $\mu_n(a) \geqq 3$.
Für Primzahlen $p > 2$ ist $a \not\equiv -2 \pmod p$ oder $a \not\equiv 2 \pmod p$ und folglich $a \equiv (a+2) - 1 - 1 \pmod{p^\alpha}$ oder $a \equiv (a-2) + 1 + 1 \pmod{p^\alpha}$ eine zulässige Darstellung von a. Und wegen $a \equiv 1 \pmod 2$ ist $a \equiv (a-2) + 1 + 1 \pmod{2^\beta}$ ein zulässige Darstellung von a. Unter Heranziehung des Lemmas erhält man somit eine zulässige Darstellung $a \equiv b_1 + b_2 + b_3 \pmod n$ von a, so daß $\mu_n(a) \leqq 3$ und folglich $\mu_n(a) = 3$.
Damit sind alle Fälle «$t \leqq n$, $n \equiv 0,1 \pmod 2$, $(a,n) \equiv 0,1 \pmod 2$» abgehandelt. Das Theorem ist bewiesen.

H. Bergmann, Hamburg

ANMERKUNGEN

[1*] Für $n \in \mathbb{N}$ ist $\varphi(n)$ die Eulersche Funktion.
[2*] Die Frage ist offensichtlich nur für endliche zyklische Gruppen interessant.
[3*] Dabei bedeutet $a \in \bar{a}$, daß $a \in \mathbb{Z}$ ein Repräsentant (Element) der Restklasse $\bar{a} \in R_n^+$ ist.
[4*] Eine Modifizierung des chinesischen Restklassensatzes.

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Didaktik und Elementarmathematik

An integral recurrence for sums of powers

Every first year calculus student encounters formulas for the sums of powers of the integers. In this note, we present an elementary proof of the curious fact that the formula for the sum of the $(k+1)$st powers can be obtained simply from the integral of the formula for the sum of the kth powers. This integral recurrence provides an easy means for computing these formulas.
We note first that if a function $F\colon [0,\infty) \to \mathbb{R}$ satisfies

$$\left.\begin{array}{ll} \text{(i)} & F(0) = 0 \quad \text{and} \\ \text{(ii)} & F(x+1) = F(x) + (x+1)^\alpha \end{array}\right\} \quad (*)$$

for some $\alpha \in \mathbb{R}$, then it follows at once that

$$F(n) = \sum_{j=1}^{n} j^\alpha$$

for every positive integer n.
We next prove the following

Lemma. *Suppose a function F satisfies condition * for some value of α. Then the new function G defined by setting*

$$G(x) = (\alpha + 1) \int_0^x F(t)\,dt + Bx$$

where

$$B = 1 - (\alpha + 1) \int_0^1 F(t)\,dt$$

*satisfies condition * for the value $\alpha + 1$.*

Proof. Certainly $G(0) = 0$. We compute

$$\begin{aligned}
G(x+1) &= (\alpha + 1) \int_0^{x+1} F(t)\,dt + B(x+1) \\
&= (\alpha + 1) \int_0^1 F(t)\,dt + (\alpha + 1) \int_1^{x+1} F(t)\,dt + Bx + B \\
&= (\alpha + 1) \int_1^{x+1} F(t)\,dt + Bx + 1 \\
&= (\alpha + 1) \int_0^x F(t+1)\,dt + Bx + 1 \\
&= (\alpha + 1) \int_0^x [F(t) + (t+1)^\alpha]\,dt + Bx + 1 \\
&= (\alpha + 1) \int_0^x F(t)\,dt + Bx + (\alpha + 1) \int_0^x (t+1)^\alpha\,dt + 1 \\
&= G(x) + (x+1)^{\alpha+1}.
\end{aligned}$$

The reasoning for $\alpha = -1$ differs from the reasoning for $\alpha \neq -1$ at the last step but in either case G satisfies the required condition.
It remains to note that if we begin our recurrence with the function $F_0(x) = x$ (which satisfies condition * for $\alpha = 0$) then we can generate polynomials $F_1, F_2, F_3, \ldots$ where

$$F_k(x) = k \int_0^x F_{k-1}(t)\,dt + B_k x$$

with

$$B_k = 1 - k \int_0^1 F_{k-1}(t)\,dt.$$

The polynomial F_k defined in this way satisfies condition * for $\alpha = k$ and consequently satisfies

$$F_k(n) = \sum_{j=1}^{n} j^k$$

for all positive integers n.

There is of course an extensive literature on these sums of powers, dating back to the Bernoullis. In [1], the authors write down explicitly a recurrence for the sequence of polynomials F_k but apparently fail to notice the simple integral expession for it. In [2] and [3] the integral recurrence is derived but the derivation depends critically on a preliminary lemma to the effect that the formulas are given by polynomials. In other references (see, for example, [4] Problems 17.20–17.29) the derivation of a formula for $\sum_{j=1}^{n} j^k$ is made somewhat more complicated by the objective of expressing the final result

$$F_k(x) = \frac{1}{k+1}[P_{k+1}(x+1) - P_{k+1}(0)].$$

in terms of Bernoulli polynomials P_k.

In contrast, the approach facilitated by our lemma has two clear advantages. First, it generalizes the result to non-integral exponents α and shows that even in the case of integral exponents $\alpha = k$, non-polynomial versions of the formula are available through different choices of the starting function $F_0(t)$ whose values for $0 < t < 1$ are at our disposal. Second, given that some result along these lines is possible, there is straight forward motivation for the formula of the lemma: consideration of asymptotic values determines the coefficient $\alpha + 1$ and then consideration of the fact that $G(1) = 1$ must hold forces the value of B.

G. Gunther, Memorial University of Newfoundland
J. B. Wilker, University of Toronto

REFERENCES

1 Budin M. A., Cantor A. J.: Simplified computation of sums of powers of integers. IEEE Transactions on Systems, Man, and Cybernetics *2*, 284 (1972).
2 Carchidi M.: Two simple recursive formulas for summing $1^k + 2^k + \ldots + n^k$. College Math. J. *18*, 406–409 (1987).
3 Levy L. S.: Summation of the series $1^n + 2^n + \ldots + x^n$ using elementary calculus. Amer. Math. Monthly *77*, 840–847 (1970).
4 Scheid F.: Schaum's Outline of Theory and Problems of Numerical Analysis. McGraw-Hill 1968.

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Aufgaben

Aufgabe 1001. Mit den Catalan-Zahlen

$$C(i) = \frac{1}{i+1}\binom{2i}{i}$$

und den Stirling-Zahlen zweiter Art

$$S(k, i) = \frac{1}{i!}\sum_{s=0}^{i}(-1)^s\binom{i}{s}(i-s)^k$$