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**Autor:** Eddy, R.H.  
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- 10 Nörlund N. E.: Differenzenrechnung. Berlin, Springer 1924.
- 11 Schroth P.: Zur Definition der Nörlundschen Hauptlösung von Differenzengleichungen. Manuscripta Math. 24, 239–251 (1978).
- 12 Schroth P.: Hauptlösungen von Differenzengleichungen. Glasnik Mat. 14, 34 (1979).

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## A generalization of Nagel's middlespoint

*In an 1836 paper, C. H. von Nagel defines the «Mittenpunkt» of a given triangle. Even though this point is readily constructed, it seems not to have found its way into the modern geometry of the triangle. In this paper, we show that by looking at the point from a slightly different point of view, one can obtain an infinite family of such points. In addition, we generalize a set of three related points.*

### 1. Introduction

In what seems to be a little-known and somewhat inaccessible paper [4], C. H. von Nagel defines the *middlespoint* (Mittenpunkt) of a given triangle  $A_1 A_2 A_3$  in the following manner.

**Definition.** Let  $S_i$ ,  $i = 1, 2, 3$ , denote the midpoints respectively of the sides  $A_{i+1} A_{i+2}$  and  $I^i$ , the excentre opposite  $A_i$  of the triangle  $A_1 A_2 A_3$ , then  $\bigcap_i S_i I^i = M$ , the middlespoint of the given triangle.

The name probably derives from the fact that the point is obtained using *middles*, i.e., centres of circles and *midpoints* of line segments. Even though this point has a simple construction using well-known concepts associated with the triangle, it seems not to appear in the available literature. One good recent paper on the subject known to us is by Baptist [1].

In this paper, we show that there exists an infinite family of such points each being a centre of perspectivity of a pair of triangles one circumscribed, the other inscribed, with respect to a given triangle. We also generalize a set of three related points referred to as «*interior middlespoint*» by Nagel. For the convenience of the reader, we supply some background.

### 2. Some properties of the middlespoint

In [5], Nagel also proves that the points  $M$ , the centroid  $S$ , and the Gergonne point  $G = \bigcap_i A_i G_i$ , where  $G_i$  is the contact point of the incircle with side  $A_{i+1} A_{i+2}$  are collinear. We add a further result in the form of a theorem which we have not previously seen.

**Theorem 1.** *The middlespoint is collinear with the orthocentre  $H$  and  $E$ , the centre of the Spieker circle.*

*Proof.* We use *trilinear* or *normal* coordinates, for a point  $P$  in the plane of  $A_1 A_2 A_3$ . In this system, the coordinate triple  $(x_1, x_2, x_3)$  (often abbreviated to  $(x_i)$ ,  $i = 1, 2, 3$ ) for  $P$  is such that  $x_i$  is proportional to the *signed* distances  $d_i$  of the given point from the side  $A_{i+1} A_{i+2}$ . The sign of  $d_i$  is positive or negative depending on whether or not  $P$  and the unit point  $I$ , the incentre, are in the same half-plane determined by  $I$  and  $A_{i+1} A_{i+2}$ . The relationship between the trilinear coordinate  $x_i$  and the actual distances  $d_i$  is given by

$$\frac{d_i}{x_i} = \frac{2\Delta}{\sum_{i=1}^3 a_i x_i}$$

where  $\Delta$  denotes the area of  $A_1 A_2 A_3$  see [6]. Since the coordinates of  $I^i$  are  $((-1)^{\delta_{ij}})$ ,  $i, j = 1, 2, 3$ , where  $\delta_{ij} = 1$  if  $i = j$  and 0 otherwise, and those for  $S$  are  $\left(\frac{1}{a_i}\right)$ , where  $a_i = \overline{A_{i+1} A_{i+2}}$ , it is an elementary exercise to show that  $M = (s - a_i)$ , where  $s = \frac{a_1 + a_2 + a_3}{2}$ , the semiperimeter of  $A_1 A_2 A_3$ . The coordinates of  $H$  are  $(\sec A_i)$  which, like  $S$ , may be found in standard sources, see for example [6]. Since we have not previously seen the coordinates for  $E$ , we provide some details. The Spieker circle is inscribed to the triangle  $S_1 S_2 S_3$  and its centre  $E$  is the midpoint of  $IN$ , where  $N$  is the Nagel point, see [4]. The actual values of  $d_i$  for  $I$  and  $N$  are  $(r)$  and  $\left(\frac{2r(s - a_i)}{a_i}\right)$  respectively and so the coordinates of  $E$  are  $\left(\frac{a_{i+1} + a_{i+2}}{a_i}\right)$ . The vanishing of the determinant

$$\begin{vmatrix} s - a_1 & s - a_2 & s - a_3 \\ \sec A_1 & \sec A_2 & \sec A_3 \\ \frac{a_2 + a_3}{a_1} & \frac{a_3 + a_1}{a_2} & \frac{a_1 + a_2}{a_3} \end{vmatrix}$$

was verified on an IBM pc using the *Reduce* compiler.

**3. A generalization.** We reconsider the definition of  $M$  and regard it from a different point of view. A point  $P$  in the interior of the triangle  $A_1 A_2 A_3$  determines an inscribed triangle  $P_1 P_2 P_3$ , where  $P_i = A_i P \cap A_{i+1} A_{i+2}$ . A second interior point  $Q$  determines a triangle,  $Q^1 Q^2 Q^3$  circumscribed about  $A_1 A_2 A_3$ , in the following manner. Consider the harmonic conjugates  $Q'_1, Q'_2, Q'_3$  of  $Q_1, Q_2, Q_3$  with respect to  $A_2$  and  $A_3$ ,  $A_3$  and  $A_1$ ,  $A_1$  and  $A_2$  which lie on a line  $q$ , the trilinear polar of  $Q$  with respect to the triangle  $A_1 A_2 A_3$ , see [2]. The point  $Q^i$  is the intersection of the lines  $A_{i+1} Q'_{i+1}$  and  $A_{i+2} Q'_{i+2}$ . If  $Q$  has trilinear coordinates  $(y_i)$ , then it is readily seen that those for  $Q^i$  are  $[(-1)^{\delta_{ij}} y_j]$ . We may now state and prove the following theorem.

**Theorem 2.** The triangles  $P_1 P_2 P_3$  and  $Q^1 Q^2 Q^3$  are perspective from the point  $T = P_i Q^i$ ,  $i = 1, 2, 3$ .

*Proof.* We begin by stating the Ceva criterion for the concurrency of the lines  $A_i X_i$  with respect to the triangle  $A_1 A_2 A_3$ , where  $X_i \in A_{i+1} A_{i+2}$ . The lines  $A_i X_i$  are concurrent at a point  $X$  if and only if

$$\prod \frac{\overline{X_i A_{i+1}}}{\overline{X_i A_{i+2}}} = -1; \quad i = 1, 2, 3,$$

and the segments are *directed* in the sense that, for example,  $\overline{XY} + \overline{YX} = 0$ . This is the *line segment criterion*. Also, we have the equivalent *angle criterion*, i.e.,  $\bigcap_i A_i X_i = X$  if and only if

$$\prod \frac{\sin(\angle X_i A_i A_{i+1})}{\sin(\angle X_i A_i A_{i+2})} = -1,$$

with angles measured in a clockwise sense taken as negative, see [4]. We proceed to show that the *line segment* criteria for the points  $P$  and  $Q$  induce, in a natural way, the *angle* criterion on the lines  $P_i Q^i$ .

From figure 1,

$$\prod \frac{\overline{P_i A_{i+1}}}{\overline{P_i A_{i+2}}} = - \prod \frac{\overline{A_{i+1} Q^i}}{\overline{A_{i+2} Q^i}} \cdot \frac{\sin \angle A_{i+1} Q^i P_i}{\sin \angle A_{i+2} Q^i P_i},$$

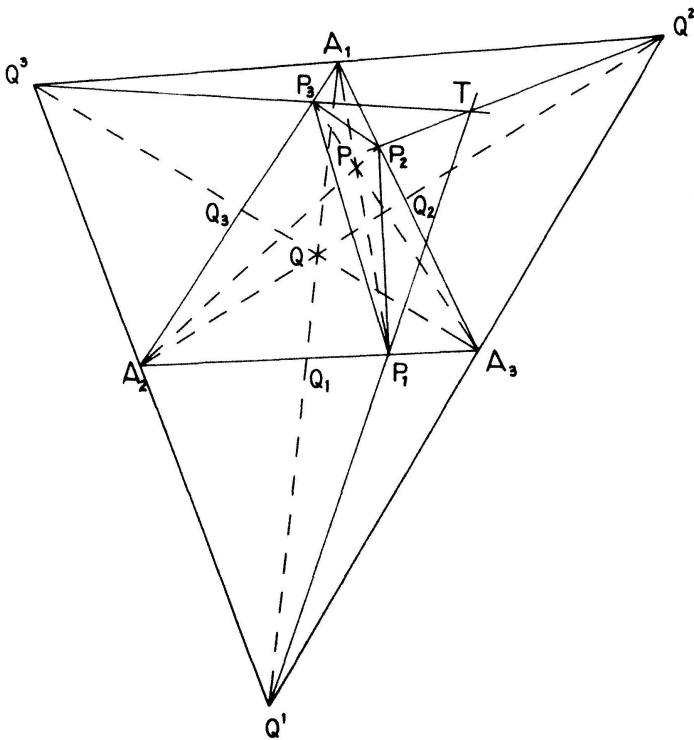


Figure 1. The generalized midpoint  $T$ .

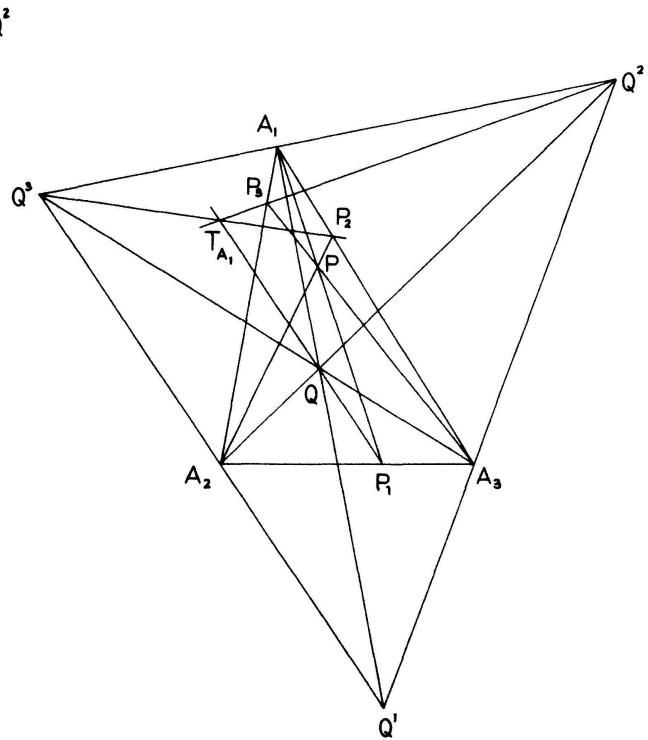


Figure 2. The generalized interior midpoint  $T_{A_1}$ .

where also the angles are directed, i.e.,  $\sin(\angle A_{i+1} P_i Q^i) + \sin(\angle A_{i+2} P_i Q^i) = 0$ . But, since the triangles  $A_1 A_2 A_3$  and  $Q^1 Q^2 Q^3$  are perspective from  $Q$ ,

$$\prod \frac{A_{i+1} Q^i}{A_{i+2} Q^i} = -1,$$

hence

$$\prod \frac{\sin(\angle A_{i+1} Q^i P_i)}{\sin(\angle A_{i+2} Q^i P_i)} = -1$$

and  $\bigcap_i P_i Q^i = T$  as claimed.

If the coordinates of  $P$  are  $(z_i)$ , then  $P_i = ((1 - \delta_{ij}) z_j)$  and so the coordinates of  $T$  are  $\left( y_i \sum_{j=1}^3 \frac{(-1)^{\delta_{ij}} y_j}{z_j} \right)$ . We leave the details of this derivation as an exercise for the reader.

In addition to  $M$ , a second interesting special case of  $T$  is found by again taking  $Q$  as the incentre and  $P$  as  $G$ , the Gergonne point of  $A_1 A_2 A_3$ , see [3]. Since, for example,  $A_2 G_1 = s - a_2$ , the coordinates of  $G$  are  $\left( \frac{1}{a_i(s - a_i)} \right)$  and those for  $T$  are  $\left( \frac{1}{s - a_i} \right)$ , which are the reciprocals of those for  $M$ . This latter point is called the *isogonal conjugate* of  $M$ , see [6], and is obtained geometrically by reflecting each of lines  $A_i M_i$  in the corresponding internal angle bisector through  $A_i$ .

#### 4. Additional points of a related type

Nagel further shows that by replacing one excentre by the incentre and interchanging the other two, one obtains three points of the form

$$M_{A_i} = S_i I \cap S_{i+1} I_{i+2} \cap S_{i+2} I_{i+1},$$

which Nagel calls the *interior* middlespoints of the given triangle. It is an elementary exercise to show that the trilinear coordinates of  $M_{A_1}$ , for example, are  $(s, s - c, s - b)$  which since all three components are positive implies that  $M_{A_1}$  is in the interior of  $A_1 A_2 A_3$ .

Each of these points may be generalized in a similar fashion as the middlespoint.

**Theorem 3.** *If, in the definition of  $T$ , we replace  $Q_i$  by  $Q$  and interchange  $Q_{i+1}$  and  $Q_{i+2}$  we obtain three points of the form*

$$T_{A_i} = P_i Q \cap P_{i+1} Q^{i+2} \cap P_{i+2} Q^{i+1}; \quad i = 1, 2, 3.$$

*Proof.* If we denote by  $\Gamma$  the product of the sines of the angles at  $Q, Q^2, Q^3$  determined by the lines  $Q P_1, Q^2 P_3, Q^3 P_2$  then

$$\prod \frac{\overline{P_i A_{i+1}}}{\overline{P_i A_{i+2}}} = - \left( \frac{\overline{A_2 Q}}{\overline{A_3 Q}} \cdot \frac{\overline{A_3 Q^3}}{\overline{A_1 Q^3}} \cdot \frac{\overline{A_1 Q^2}}{\overline{A_2 Q^2}} \right) \cdot \Gamma.$$

By inspecting figure 2 and considering the relationships

$$\frac{\overline{A_2 Q}}{\overline{A_3 Q}} = \frac{\overline{A_2 Q^3} \sin(\angle A_3 Q^3 A_2)}{\overline{A_3 Q^2} \sin(\angle A_3 Q^2 A_2)}, \quad \frac{\overline{A_3 Q^3}}{\overline{A_2 Q^2}} = \frac{\sin(\angle Q^3 A_2 A_3)}{\sin(\angle A_3 Q^3 A_2)} \cdot \frac{\sin(\angle A_3 Q^2 A_2)}{\sin(\angle Q^2 A_3 A_2)},$$

$$\frac{\overline{A_3 Q^1}}{\overline{A_2 Q^1}} = \frac{\sin(\angle A_3 A_2 Q^3)}{\sin(\angle A_2 A_3 Q^2)},$$

we see that the bracketed product is equal to

$$\prod \frac{\overline{A_i Q^{i+1}}}{\overline{A_i Q^{i+2}}} = -1,$$

hence  $\Gamma = -1$  and the lines  $P_1 Q$ ,  $P_2 Q_3$ ,  $P_3 Q_2$  are concurrent at  $T_{A_1}$ . Similar arguments hold for  $T_{A_2}$  and  $T_{A_3}$ .

The coordinates for  $T_{A_i}$  may be obtained directly or from those for  $T$  by noting the relevant differences in the construction of the required point. For example, the coordinates of  $T_{A_1}$  are given by

$$x_1 = y_1 \left( \frac{y_1}{z_1} + \frac{y_2}{z_2} + \frac{y_3}{z_3} \right),$$

$$x_2 = y_2 \left( \frac{y_1}{z_1} + \frac{y_2}{z_2} - \frac{y_3}{z_3} \right),$$

$$x_3 = y_3 \left( \frac{y_1}{z_1} - \frac{y_2}{z_2} + \frac{y_3}{z_3} \right).$$

As an example, we see that the coordinates for the interior midpoint  $M_{A_1}$  are as given previously with similar representations for  $M_{A_2}$  and  $M_{A_3}$ . In addition, it is easy to see that  $T_{A_i}$  is not necessarily in the interior of the triangle  $A_1 A_2 A_3$ .

R. H. Eddy, Dept. of Mathematics and Statistics,  
Memorial University of NF, St. John's, Newfoundland

#### REFERENCES

- 1 Baptist P.: Über Nagelsche Punktepaare, Mathematische Semesterberichte. Band XXXV, pp. 118–126 (1988).
- 2 Coxeter H. S. M.: The Real Projective Plane, 2nd Ed., Cambridge University Press, 1961.
- 3 Groenman J. T.: Problem 1272, solution by Jordi Dou, Crux Math. 14, 256 (1988).
- 4 Johnson R. A.: Advanced Euclidean Geometry (Modern Geometry), pp. 145–147, 225–227, Dover, New York 1969.
- 5 Nagel C. H.: Untersuchungen über die wichtigsten zum Dreieck gehörenden Kreise, Leipzig 1836.
- 6 Sommerville D. M. Y.: Analytical Conics, pp. 159–160, G. Bell and Sons, Ltd., London 1961.