

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 45 (1990)
Heft: 1

Artikel: Extension theorems for integral representations of solutions of a functional equation
Autor: Cannizzo, A.
DOI: <https://doi.org/10.5169/seals-42405>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 18.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Extension theorems for integral representations of solutions of a functional equation [*]

Summary. Integral representations $F(x) = \int_0^\infty e^{-t} \Phi(t, x-1) dt$ of solutions of the functional equation $F(x+1) = g(x)F(x)$, which were obtained in previous papers for the interval $(0, 1)$, are extended to solutions on \mathbb{R}_+ . Additional properties of the kernel function Φ are discussed in this context.

1. Introduction

We start with the following observations:

a) Let $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}$, $g: \mathbb{R}_+ \rightarrow \mathbb{R}$ be given. If $g = \exp \varphi$, then the solution set of the difference equation

$$G(x+1) - G(x) = \varphi(x) \tag{0}$$

is known, iff the set of positive solutions of

$$F(x+1) = g(x)F(x) \tag{1}$$

is known. In general, equations (0) and (1) are not equivalent.

b) The theory of equation (0), and in particular the theory of its *principal solutions* in the sense of Nörlund, Krull, Schroth (see [4], and [7] to [12]), is well developed.

There is no obvious way to extend this concept of a principal solution to the general case of equation (1).

c) In the theory of equation (1) solutions of the form

$$F(x) = \int_0^\infty e^{-t} \Phi(t, x-1) dt \tag{2}$$

play a prominent role. Take as an example $F(x) = \Gamma(x)$, Euler's Gamma function.

In [1], [5] and [6] we obtained solutions of (1) which are of type (2), where the kernel satisfies the functional-differential equation

$$\Phi_t(t, x) = g(x) \Phi(t, x-1). \tag{3}$$

For convenience of the reader we restate Theorem 1 of [1] in a slightly modified form:

Theorem 1. Let $U \subset \mathbb{R}$ and $g: U \rightarrow \mathbb{R}$. Assume that $\Phi: \mathbb{R}_+ \times (U \cup \{u-1: u \in U\}) \rightarrow \mathbb{R}$ has the following properties:

$h_1)$ $\Phi_t(t, x)$ exists for every $t \in \mathbb{R}_+$ and $x \in U$,

[*] Supported by Ministero Pubblica Istruzione, Rome (Italy).

$h_2)$ $\lim_{t \rightarrow 0} \Phi(t, x) = 0$ for every $x \in \mathbb{U}$,

$h_3)$ for every $x \in \mathbb{U}$ there are $p \in (0, 1)$, $t_0 \in \mathbb{R}_+$, $K \in \mathbb{R}_+$ such that

$$e^{-pt} |\Phi(t, x)| \leq K \quad \text{for every } t \geq t_0.$$

If Φ satisfies the equation (3) for $t \in \mathbb{R}_+$, $x \in \mathbb{U}$ then the integrals

$$F(x) = \int_0^{\infty} e^{-t} \Phi(t, x-1) dt \quad \text{and} \quad F(x+1) = \int_0^{\infty} e^{-t} \Phi(t, x) dt$$

exist for every $x \in \mathbb{U}$ and equation (1) holds for $x \in \mathbb{U}$.

In [1] we used instead of $h_3)$: $\lim_{t \rightarrow +\infty} e^{-pt} \Phi(t, x) = 0$ for every $x \in \mathbb{U}$ and every $p \in (0, 1]$. It

is readily checked that the theorem remains true with the weaker condition $h_3)$.

In the following Theorem 2 of [2] we introduced another functional equation for the kernel Φ , motivated by the reflection formula for principal solutions:

$$\Phi(t, -x) = 1/\Phi(t, x). \quad (4)$$

Theorem 2. Let $\mathbb{U} = (0, 1)$, $g: \mathbb{U} \rightarrow \mathbb{R}_+$, $\hat{\Phi}: \mathbb{R}_+ \times ((-1, 0) \cup (0, 1)) \rightarrow \mathbb{R}_+$ and assume that $\hat{\Phi}$ satisfies (4) for $t \in \mathbb{R}_+$, $x \in \mathbb{U}$. Then $\hat{\Phi}$ satisfies $h_1)$, $h_2)$ and $h_3)$ of Theorem 1 and (3) for $t \in \mathbb{R}_+$, $x \in \mathbb{U}$, iff

$$\hat{\Phi}(t, x) = h(x) \hat{\gamma}(x)^{-\hat{\gamma}(x)} t^{\hat{\gamma}(x)}, \quad t \in \mathbb{R}_+, x \in \mathbb{U},$$

where $\hat{\gamma}(x) := g(x)/[g(1-x) + g(x)]$ and $h: \mathbb{U} \rightarrow \mathbb{R}_+$ is solution of the reflection equation

$$h(x)h(1-x) = g(x)^{\hat{\gamma}(x)} g(1-x)^{\hat{\gamma}(1-x)}.$$

This theorem shows that a kernel $\hat{\Phi}$ which satisfies (3) and (4) and some regularity conditions must be of separation type: $\hat{\Phi}(t, x) = \hat{G}(x) t^{\hat{\gamma}(x)}$ (Functions which will be later subject to an extension procedure are marked with a circumflex). The aim of this note is to extend these results for kernels defined on $\mathbb{R}_+ \times (-1, \infty)$ and to study the behaviour of kernels of separation type with respect to the equations (1), (2), (3), (4).

2. Structure and Extension Theorems

We begin this section with a remark on kernels of the type $\Phi(t, x) = G(x) t^{\gamma(x)}$ and their connection with the functional equation (1). Euler's gamma function is involved.

Theorem 3. Assume $g: \mathbb{R}_+ \rightarrow \mathbb{R}$, $G: (-1, \infty) \rightarrow \mathbb{R}$, $\gamma: (-1, \infty) \rightarrow (-1, \infty)$ and

$$F(x) = \int_0^{\infty} e^{-t} G(x-1) t^{\gamma(x-1)} dt, \quad x \in \mathbb{R}_+.$$

Then the following conditions are equivalent:

- (i) F satisfies (1) for $x \in \mathbb{R}_+$,
- (ii) $G(x) \Gamma(\gamma(x) + 1) = g(x) G(x - 1) \Gamma(\gamma(x - 1) + 1)$ for $x \in \mathbb{R}_+$.

Proof. For $x \in \mathbb{R}_+$, the following equations are equivalent:

$$\int_0^\infty e^{-t} G(x) t^{\gamma(x)} dt = g(x) \int_0^\infty e^{-t} G(x - 1) t^{\gamma(x-1)} dt,$$

$$G(x) \int_0^\infty e^{-t} t^{\gamma(x)} dt = g(x) G(x - 1) \int_0^\infty e^{-t} t^{\gamma(x-1)} dt$$

$$G(x) \Gamma(\gamma(x) + 1) = g(x) G(x - 1) \Gamma(\gamma(x - 1) + 1). \quad \blacksquare$$

The solutions of (3) which are of separation type now are characterized in the following

Theorem 4. Assume $g: \mathbb{R}_+ \rightarrow \mathbb{R}$, $G: (-1, \infty) \rightarrow \mathbb{R}$, $\gamma: (-1, \infty) \rightarrow \mathbb{R}$, and $\gamma(x) G(x) \neq 0$ for every $x \in \mathbb{R}_+$. Then the following statements are equivalent:

- (j) $\Phi(t, x) = G(x) t^{\gamma(x)}$ satisfies (3) for $t \in \mathbb{R}_+$, $x \in \mathbb{R}_+$,
- (jj) $\gamma(x - 1) + 1 = \gamma(x)$ and $G(x) \gamma(x) = g(x) G(x - 1)$ for $x \in \mathbb{R}_+$.

Proof. From (j) we deduce

$$\gamma(x) G(x) t^{\gamma(x)-1} = g(x) G(x - 1) t^{\gamma(x-1)}, \quad x \in \mathbb{R}_+.$$

With $t = 1$ we get the second equation of (jj). Now $\gamma(x) G(x) \neq 0$ implies the first equation of (jj). Conversely, if $\Phi(t, x) = G(x) t^{\gamma(x)}$, the equations (jj) yield

$$\Phi_t(t, x) = \gamma(x) G(x) t^{\gamma(x)-1} = g(x) G(x - 1) t^{\gamma(x-1)} = g(x) \Phi(t, x - 1).$$

Here the assumption $\gamma(x) G(x) \neq 0$ was not used. \blacksquare

Now we show that a kernel

$$\hat{\Phi}(t, x) = \hat{G}(x) t^{\hat{\gamma}(x)} \quad t \in \mathbb{R}_+, x \in [0, 1),$$

can be extended to

$$\Phi(t, x) = G(x) t^{\gamma(x)} \quad t \in \mathbb{R}_+, x \in (-1, \infty)$$

a solution of (3). More precisely:

Theorem 5. Let $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\hat{\gamma}: [0, 1) \rightarrow (-1, \infty)$, $\hat{G}: [0, 1) \rightarrow \mathbb{R}$, and assume

$$\hat{\Phi}(t, x) = \hat{G}(x) t^{\hat{\gamma}(x)} \quad t \in \mathbb{R}_+, x \in [0, 1).$$

Then there are extensions $\gamma: (-1, \infty) \rightarrow (-1, \infty)$, $G: (-1, \infty) \rightarrow \mathbb{R}$, $\Phi: \mathbb{R}_+ \times (-1, \infty) \rightarrow \mathbb{R}_+$ of $\hat{\gamma}$, \hat{G} , $\hat{\Phi}$ such that $\Phi(t, x) = G(x)t^{\gamma(x)}$ satisfies (3) for $t \in \mathbb{R}_+$, $x \in \mathbb{R}_+$.
 If $\hat{\gamma}(x)\hat{G}(x) \neq 0$ for every $x \in [0, 1)$ and $\hat{c}_r(0) \neq 0$ then Φ is uniquely determined.

Proof. For the existence statement define $\gamma(x) := \hat{\gamma}(x - [x]) + [x]$ for $x \in (-1, \infty)$ and $G(x) := \hat{G}(x + 1)\hat{\gamma}(x + 1)/g(x + 1)$ for $x \in (-1, 0)$, $G(x) := \hat{G}(x - [x]) \prod_{k=0}^{[x]} \frac{g(x - k + 1)}{\gamma(x) + k - [x]}$ for $x \geq 0$, with the usual convention about an empty product; $[x]$ denotes the greatest integer not exceeding x . Then γ and G are extensions of $\hat{\gamma}$ and \hat{G} which satisfy equations (jj) of Theorem 4:

$$\begin{aligned} \gamma(x - 1) + 1 &= \hat{\gamma}(x - 1 - [x - 1]) + [x - 1] + 1 = \hat{\gamma}(x - [x]) + [x] = \gamma(x), \\ G(x)\gamma(x) &= \hat{G}(x)\hat{\gamma}(x) = G(x - 1)g(x) \quad \text{for } x \in [0, 1), \\ G(x)\gamma(x) &= [\hat{G}(x - 1)g(x)/\gamma(x)]\gamma(x) = G(x - 1)g(x) \quad \text{for } x \in [1, 2), \\ G(x)\gamma(x) &= [\hat{G}(x - 2)g(x)g(x - 1)/(\gamma(x) - 1)\gamma(x)]\gamma(x) \\ &= [\hat{G}(x - 2)g(x - 1)/\gamma(x - 1)]g(x) = G(x - 1)g(x) \quad \text{for } x \in [2, 3). \end{aligned}$$

etc. . . .

Because of $\gamma(x) - [x] = \hat{\gamma}(x - [x])$ the denominators $\gamma(x) + k - [x]$ in the above products are positive when $[x] \geq 1$.

The final remark in the proof of Theorem 4 shows that $\Phi(t, x) = G(x)t^{\gamma(x)}$ is a solution of (3) for $t \in \mathbb{R}_+$, $x \in \mathbb{R}_+$, regardless if $\gamma(x)G(x) \neq 0$ for $x \in [0, 1)$ or not.

According to the uniqueness statement we note for any $\Phi(t, x) = G(x)t^{\gamma(x)}$ satisfying (3), necessarily

$$\gamma(x)G(x)t^{\gamma(x)-1} = g(x)G(x-1)t^{\gamma(x-1)}, \quad t \in \mathbb{R}_+, x \in \mathbb{R}_+.$$

Hence

$$\gamma(x)G(x) = g(x)G(x-1) \quad \text{for } x \in \mathbb{R}_+.$$

Now our assumption $\hat{\gamma}(x)\hat{G}(x) \neq 0$ for $x \in [0, 1)$ implies: $\gamma(x - 1) = \gamma(x) - 1$ for $x \in (0, 1)$. For $x \in [1, 2)$ we have $\gamma(x)G(x) = g(x)G(x - 1) \neq 0$ because of $g(x) > 0$ and $G(x - 1) \neq 0$ and again $\gamma(x - 1) = \gamma(x) - 1$. By induction $\gamma(x - 1) = \gamma(x) - 1$ for every $x \in \mathbb{R}_+$. Consequently in this case the extensions γ, G of $\hat{\gamma}, \hat{G}$ must be defined as given above: the function γ defined above is the unique solution of $\gamma(x) - \gamma(x - 1) = 1$ which coincides on $[0, 1)$ with $\hat{\gamma}$ and the function G defined above is the unique solution of $\gamma(x)G(x) = g(x)G(x - 1)$ which coincides on $[0, 1)$ with \hat{G} . ■

Finally we apply the previous results to extend the kernel $\hat{\Phi}$ which was obtained in Theorem 2 for $x \in (-1, 0) \cup (0, 1)$ to a kernel Φ , defined for $x \in (-1, \infty)$ under preserving at all its useful properties:

Theorem 6. Let $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be given and assume $\hat{\Phi}: \mathbb{R}_+ \times ((-1, 0) \cup (0, 1)) \rightarrow \mathbb{R}_+$ satisfies conditions $h_1), h_2), h_3)$ of Theorem 1 and equations (3) and (4) for $t \in \mathbb{R}_+$, $x \in (0, 1)$. Then

there is an extension $\Phi: \mathbb{R}_+ \times (-1, \infty) \rightarrow \mathbb{R}_+$ of $\hat{\Phi}$, which is of separation type and satisfies (3) for $t \in \mathbb{R}_+, x \in \mathbb{R}_+$.

The function $F(x) = \int_0^\infty e^{-t} \Phi(t, x-1) dt$ is a solution of (1) on \mathbb{R}_+ .

Proof. By Theorem 2, $\hat{\Phi}(t, x) = h(x) \hat{\gamma}(x)^{-\hat{\gamma}(x)} t^{\hat{\gamma}(x)}$, for $t \in \mathbb{R}_+, x \in (0, 1)$, where $\hat{\gamma}(x) = g(x)/[g(1-x) + g(x)]$ and h is a positive solution of the reflection equation

$$h(x)h(1-x) = g(x)^{\hat{\gamma}(x)}g(1-x)^{\hat{\gamma}(x-1)}, \quad x \in (0, 1).$$

Define $\hat{G}(x) := h(x) \hat{\gamma}(x)^{-\hat{\gamma}(x)}$ for $x \in (0, 1)$ and $\hat{G}(0) := \alpha, \hat{\gamma}(0) := \beta$ with $\alpha > 0, \beta \geq 0$ and $\hat{\Phi}(t, 0) := \alpha t^\beta$. Then $\hat{G}: [0, 1) \rightarrow \mathbb{R}_+$ and $\hat{\gamma}: [0, 1) \rightarrow \mathbb{R}_+$.

By Theorem 5 there is a unique extension $\Phi: \mathbb{R}_+ \times (-1, \infty) \rightarrow \mathbb{R}_+$ of $\hat{\Phi}$ such that $\Phi(t, x) = G(x)t^{\gamma(x)}$, G and γ extensions of \hat{G} and $\hat{\gamma}$ as defined in the proof of Theorem 5. Φ satisfies (3) for every $t \in \mathbb{R}_+$ and every $x \in \mathbb{R}_+$.

The conditions $h_1), h_2), h_3)$ of Theorem 1 in the case $\mathbb{U} = \mathbb{R}_+$ are satisfied for our kernel Φ , hence

$$F(x) = \int_0^\infty e^{-t} \Phi(t, x-1) dt$$

defines a solution of (1) on \mathbb{R}_+ .

Note that every collection $\{h, \alpha, \beta\}$, h a positive solution of the reflection formula above, $\alpha > 0, \beta \geq 0$, yields a kernel Φ ; also note that the reflection equation has always a positive solution:

$$h(x) = g(x)^{\hat{\gamma}(x)}. \quad \blacksquare$$

A. Cannizzo, Dipartimento di Matematica e Applicazioni, Palermo

REFERENCES

- 1 Amaducci T., Cannizzo A.: Soluzioni integrali dell'equazione funzionale $t(y+1) = f(y)t(y)$. Boll. Un. Mat. Ital. 5-A, 39-45 (1986).
- 2 Anastassiadis J.: Définition des fonctions Eulériennes par des équations fonctionnelles. Paris, Gauthier-Villars 1964.
- 3 Artin E.: The Gamma Function. Holt, Rinehart and Winston, New York 1964.
- 4 Busing L.: Vergleich von Hauptlösungsbegriffen für Nörlundsche Differenzgleichungen und Anwendungen auf die von Bendersky untersuchten Gammafunktionen (Dissertation). Clausthal 1982.
- 5 Cannizzo A.: Rappresentazione Euleriana estesa delle soluzioni dell'equazione di Artin. Rend. Sem. Mat. Univers. Politecn. Torino 44, 293-302 (1986).
- 6 Cannizzo A., Rodonò L.: Soluzioni dell'equazione di Artin in forma Euleriana estesa. Istituto Lombardo (Rend. Sc.) A 119, 51-64 (1985).
- 7 Kairies H.-H.: Einbettungen von $\log \Gamma$ durch Scharen spezieller Krull-Normallösungen. Aequationes Math. 29, 28-35 (1985).
- 8 Krull W.: Bemerkungen zur Differenzgleichung $g(x+1) - g(x) = \varphi(x)$. Math. Nachr. 1, 365-376 (1948) und 2, 251-262 (1949).
- 9 Kuczma M.: Functional equations in a single variable. Polish Scientific Publishers, Warszawa 1968.

- 10 Nörlund N. E.: Differenzenrechnung. Berlin, Springer 1924.
 11 Schroth P.: Zur Definition der Nörlundschen Hauptlösung von Differenzgleichungen. Manuscripta Math. 24, 239–251 (1978).
 12 Schroth P.: Hauptlösungen von Differenzgleichungen. Glasnik Mat. 14, 34 (1979).

© 1990 Birkhäuser Verlag, Basel

0013-6018/90/010009-06\$1.50 + 0.20/0

A generalization of Nagel's middlespoint

In an 1836 paper, C. H. von Nagel defines the «Mittelpunkt» of a given triangle. Even though this point is readily constructed, it seems not to have found its way into the modern geometry of the triangle. In this paper, we show that by looking at the point from a slightly different point of view, one can obtain an infinite family of such points. In addition, we generalize a set of three related points.

1. Introduction

In what seems to be a little-known and somewhat inaccessible paper [4], C. H. von Nagel defines the *middlespoint* (Mittelpunkt) of a given triangle $A_1 A_2 A_3$ in the following manner.

Definition. Let S_i , $i = 1, 2, 3$, denote the midpoints respectively of the sides $A_{i+1} A_{i+2}$ and I^i , the excentre opposite A_i of the triangle $A_1 A_2 A_3$, then $\bigcap_i S_i I^i = M$, the middlespoint of the given triangle.

The name probably derives from the fact that the point is obtained using *middles*, i.e., centres of circles and *midpoints* of line segments. Even though this point has a simple construction using well-known concepts associated with the triangle, it seems not to appear in the available literature. One good recent paper on the subject known to us is by Baptist [1].

In this paper, we show that there exists an infinite family of such points each being a centre of perspectivity of a pair of triangles one circumscribed, the other inscribed, with respect to a given triangle. We also generalize a set of three related points referred to as «*interior middlespoint*» by Nagel. For the convenience of the reader, we supply some background.

2. Some properties of the middlespoint

In [5], Nagel also proves that the points M , the centroid S , and the Gergonne point $G = \bigcap_i A_i G_i$, where G_i is the contact point of the incircle with side $A_{i+1} A_{i+2}$ are collinear. We add a further result in the form of a theorem which we have not previously seen.