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Convexity and subharmonicity

This article draws together a number of results (some recent) which link notions of convexity with subharmonic functions. No specialist knowledge is assumed and all proofs are elementary in nature.

1. Subharmonic functions

We shall be concerned with Euclidean space $\boldsymbol{R}^n$ $(n \geq 2)$, points of which are denoted by $X = (x_1, \ldots, x_n)$. We write $|X| = (x_1^2 + \ldots + x_n^2)^{1/2}$, and denote the open ball of radius r centred at X by $B(X,r)$. The closure and boundary of a subset E of $\boldsymbol{R}^n$ will be denoted respectively by $\bar{E}$ and ∂E.

Recall that a function u on an open subset ω of $\boldsymbol{R}^n$ is called *harmonic* on ω if it is twice continuously differentiable and satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \equiv 0.$$

(This equation arises naturally in gravitation, electrostatics, hydrodynamics and the theory of analytic functions.) Alternatively, letting $M(u; X, r)$ denote the mean value of u over the sphere $\partial B(X,r)$ whenever $\overline{B(X,r)} \subset \omega$, a function u is harmonic on ω if and only if:

(i) $-\infty < u < +\infty$ on ω;
(ii) u is continuous on ω; and
(iii) $\overline{B(X,r)} \subset \omega \Rightarrow u(X) = M(u; X, r)$.

By subdividing (i)–(iii) above we arrive at the dual notions of sub- and superharmonicity (due to F. Riesz [4]). Thus a function u on ω is called *subharmonic* if:

(ia) $-\infty \leq u < +\infty$ on ω [$u \not\equiv -\infty$ on any component of ω];
(iia) u is upper semicontinuous (u.s.c.), i.e. $\{X \in \omega : u(X) < c\}$ is open for any $c \in \boldsymbol{R}$;
(iiia) $\overline{B(X,r)} \subset \omega \Rightarrow u(X) \leq M(u; X, r)$.

A function u on ω is called *superharmonic* if:

(ib) $-\infty < u \leq +\infty$ on ω [$u \not\equiv +\infty$ on any component of ω];
(iib) u is lower semicontinuous, i.e. $\{X \in \omega : u(X) > c\}$ is open for any $c \in \boldsymbol{R}$;
(iiib) $\overline{B(X,r)} \subset \omega \Leftarrow u(X) \geq M(u; X, r)$.

Such functions have many applications. For example, if f is analytic in $\boldsymbol{C}$ and $f \not\equiv 0$, then $\log|f|$ is subharmonic. Again, the gravitational potential energy due to a mass distribution is superharmonic on $\boldsymbol{R}^3$. The following observations are immediate:

(I) u is subharmonic if and only if $-u$ is superharmonic;
(II) u is harmonic if and only if both u and $-u$ are subharmonic;
(III) if u and v are subharmonic and $a, b \geq 0$, then $au + bv$ is subharmonic.

An equivalent formulation of the definition of a subharmonic function is obtained if we replace (iii a) above by:

(iii a′) for any open set W with compact closure in ω, and for any continuous function h on $\bar{W}$ which is harmonic on W and satisfies $h \geq u$ on ∂W, we have $h \geq u$ on W.

It is this condition which accounts for the name *sub*harmonic.
We conclude this section by interpreting the above definitions for functions of one real variable. Laplace's equation for the real line is simply $d^2u/dx^2 \equiv 0$, so that "harmonic" functions are just linear functions of the form $ax + b$ ($a, b \in \boldsymbol{R}$). In view of (iii a′) above, a function f on an interval is "subharmonic" if, whenever $f(x) \leq ax + b$ for $x = x_1, x_2$, the inequality remains valid for x between x_1 and x_2. In other words, "subharmonic" means "convex" when applied to functions on the real line.
Thus subharmonic functions can be regarded as a generalization to higher dimensions of convex functions. This explains (at least in part) why notions of convexity recur frequently in the study of subharmonic functions.

2. Composition properties

If we begin with functions of one real variable, we can make the following simple observations concerning compositions of functions:

$$[\text{Convex}] \circ [\text{Linear}] = [\text{Convex}]$$
$$[\text{Increasing Convex}] \circ [\text{Convex}] = [\text{Convex}].$$

("Increasing" is to be interpreted in the wide sense, i.e. non-decreasing). It is an easy consequence of Jensen's inequality that these properties carry across to higher dimensions as follows:

$$[\text{Convex}] \circ [\text{Harmonic}] = [\text{Subharmonic}] \tag{1}$$
$$[\text{Increasing Convex}] \circ [\text{Subharmonic}] = [\text{Subharmonic}]. \tag{2}$$

However, much more can be said. In what follows we interpret $\phi(-\infty)$ as $\lim_{x \to -\infty} \phi(x)$.

Theorem 1. *The function $v\phi(u/v)$ is subharmonic in each of the following cases:*

(i) u is harmonic, v is positive and harmonic, ϕ is convex;
(ii) u is subharmonic, v is positive and harmonic, ϕ is convex and increasing;
(iii) u is subharmonic, v is positive and superharmonic, ϕ is convex, increasing, and $\phi(x) = 0$ for $x \leq 0$.

By taking $v \equiv 1$, it is clear that (i) and (ii) include (1) and (2) above. The proof relies on a simple lemma.

Lemma 1. *If $\{u_\alpha\}$ is a family of subharmonic functions on ω and $\sup_\alpha u_\alpha$ is u.s.c. and less than $+\infty$, then $\sup_\alpha u_\alpha$ is subharmonic on ω.*

Proof of Lemma.

$$\overline{B(X,r)} \subset \omega \Rightarrow u_\beta(X) \leq M(u_\beta; X, r) \leq M(\sup_\alpha u_\alpha; X, r)$$
$$\Rightarrow \sup_\alpha u_\alpha \leq M(\sup_\alpha u_\alpha; X, r).$$

Thus $\sup_\alpha u_\alpha$ satisfies the conditions (i a)–(iii a) of Section 1.

Proof of Theorem. Corresponding to each part of the theorem, ϕ can be written as:

(i) $\phi(x) = \sup\{ax + b : a, b \in \boldsymbol{R}$ such that $at + b \leq \phi(t)\ \forall t \in \boldsymbol{R}\}$;
(ii) $\phi(x) = \sup\{ax + b : a \geq 0$ and $b \in \boldsymbol{R}$ such that $at + b \leq \phi(t)\ \forall t \in \boldsymbol{R}\}$;
(iii) $\phi(x) = \sup\{ax + b : a \geq 0$ and $b \leq 0$ such that $at + b \leq \phi(t)\ \forall t \in \boldsymbol{R}\}$.

Thus $v\,\phi(u/v)$ can be written as

$$\sup_{a,b} v[a(u/v) + b] = \sup_{a,b}[a u + b v]$$

and $au + bv$ is subharmonic for the appropriate values of a, b in each of the three cases. Theorem 1 will follow from Lemma 1 if we can show that $v\,\phi(u/v)$ is u.s.c. This is clear for (i) as u, v and ϕ are all continuous. For part (ii) u is u.s.c. and v is continuous, so

$$\{X : u(X)/v(X) < c\} = \bigcup_{d \in \boldsymbol{R}} [\{X : u(X) < d\} \cap \{X : d < c\,v(X)\}], \tag{3}$$

which is open for any $c \in \boldsymbol{R}$. Thus u/v is u.s.c. Since ϕ is continuous and increasing, we can add that $\phi(u/v)$ is also u.s.c. Reasoning again as in (3) we see that $v\,\phi(u/v)$ is u.s.c. as required. A similar argument disposes of (iii), so the Theorem is proved.
Theorem 1 and its proof transfer easily to the axiomatic setting of harmonic spaces and so can be applied to subsolutions of a wide class of elliptic and parabolic p.d.e.'s (see [2]). This is particularly interesting because (1) and (2) do not hold for harmonic spaces, the reason being that the constant function 1 is not necessarily harmonic in the general setting.

3. Spherical means

If we take a suitable summary of the values of a subharmonic function over a sphere of fixed centre and radius r, convex functions reappear. The simplest example of this is the following well known analogue of Hadamard's Three Circles Theorem. Let O be the origin of $\boldsymbol{R}^n$, let $\psi_2(x) = \log x$ and $\psi_n(x) = x^{2-n}$ $(n \geq 3)$.

Theorem 2. *If u is subharmonic in $B(O,R)$, then the supremum of u over $\partial B(O,r)$, denoted by $N(u,r)$, is convex as a function of $\psi_n(r)$ for $0<r<R$.*

Proof. Let $0<r_1<r<r_2<R$, and choose a,b such that $N(u,r_i)\le a\psi_n(r_i)+b$ for $i=1,2$. We want to deduce $N(u,r)\le a\psi_n(r)+b$. Now

$$u(X)\le a\psi_n(|X|)+b \tag{4}$$

on the boundary of the annulus $\{X: r_1<|X|<r_2\}$. Since (as is verified by direct differentiation) $\psi_n(|X|)$ is harmonic in $\boldsymbol{R}^n\backslash\{O\}$, condition (iii a') shows that (4) remains valid for $|X|=r$. Thus $N(u,r)\le a\psi_n(r)+b$ as required.

Riesz [4] proved the same convexity property for the integral mean $M(u;O,r)$. In fact, the conclusion of Theorem 2 holds for $\log M(e^u;O,r)$ and (provided $u\ge 0$) $\{M(u^p;O,r)\}^{1/p}$, $p>1$. Behind these results lies a form of Minkowski's inequality which is particularly relevant to subharmonic functions, as will become apparent below. In what follows ϕ is a continuous function on an interval $I\subseteq[-\infty,+\infty)$, twice continuously differentiable on the interior I^0 of I, and u takes values only in I.

Theorem 3. *Let u be subharmonic in $B(O,R)$, let $\phi'>0$, $\phi''>0$ and let ϕ'/ϕ'' be concave on I^0. Then $\phi^{-1}\{M(\phi\circ u;O,r)\}$ is convex as a function of $\psi_n(r)$ for $0<r<R$.*

This result is due to Solomentsev [5] but we will give a different argument. Observe that e^x on $[-\infty,+\infty)$, $x^p\,(p>1)$ and $\cosh x$ on $[0,+\infty)$ satisfy the hypotheses of the theorem.

Proof. To simplify notation, we give the proof for $\boldsymbol{R}^2$ which we identify with $\boldsymbol{C}$. If f is an u.s.c. function on $[0,2\pi]^2$ taking values only in I, and ϕ satisfies the hypotheses of the theorem, then

$$\phi^{-1}\left\{\int_{[0,2\pi]}\phi\left[\int_{[0,2\pi]} f(\theta_1,\theta_2)\frac{d\theta_2}{2\pi}\right]\frac{d\theta_1}{2\pi}\right\}\le\int_{[0,2\pi]}\phi^{-1}\left\{\int_{[0,2\pi]}\phi\circ f(\theta_1,\theta_2)\frac{d\theta_1}{2\pi}\right\}\frac{d\theta_2}{2\pi}. \tag{5}$$

(This is an integral form of an inequality in Hardy, Littlewood and Pólya [3; § 3.16].) Let

$$s(re^{i\theta})=\phi^{-1}\{M(\phi\circ u;O,r)\}\quad(0<r<R),$$

and $0<\varrho<R-r$. Using (iii a) (applied to u) and then (5) we have

$$\begin{aligned}
s(re^{i\theta})&=\phi^{-1}\left\{\int_{[0,2\pi]}\phi\circ u(re^{i(\theta+\theta_1)})\frac{d\theta_1}{2\pi}\right\}\\
&\le\phi^{-1}\left\{\int_{[0,2\pi]}\phi\left[\int_{[0,2\pi]}u(re^{i(\theta+\theta_1)}+\varrho e^{i(\theta_2+\theta_1)})\frac{d\theta_2}{2\pi}\right]\frac{d\theta_1}{2\pi}\right\}\\
&\le\int_{[0,2\pi]}\phi^{-1}\left\{\int_{[0,2\pi]}\phi\circ u([re^{i\theta}+\varrho e^{i\theta_2}]e^{i\theta_1})\frac{d\theta_1}{2\pi}\right\}\frac{d\theta_2}{2\pi}\\
&=\int_{[0,2\pi]}s(re^{i\theta}+\varrho e^{i\theta_2})\frac{d\theta_2}{2\pi}\\
&=M(s;re^{i\theta},\varrho).
\end{aligned}$$

Thus s satisfies property (iii a). It is straightforward to check that a function f on ω is u.s.c. (as defined in (ii a)) if and only if $\limsup_{Y \to X} f(Y) \le f(X)$ for any $X \in \omega$. Now

$$\limsup_{r \to r_0} \int_{[0,2\pi]} \phi \circ u(r e^{i\theta}) \frac{d\theta}{2\pi} \le \int_{[0,2\pi]} \limsup_{r \to r_0} \phi \circ u(r e^{i\theta}) \frac{d\theta}{2\pi}$$

by Fatou's Lemma (u.s.c. functions which do not take the value $+\infty$ are readily seen to be locally bounded above), so the u.s. continuity of $\phi \circ u$ gives

$$\limsup_{r \to r_0} M(\phi \circ u; O, r) \le \int_{[0,2\pi]} \phi \circ u(r_0 e^{i\theta}) \frac{d\theta}{2\pi} = M(\phi \circ u; O, r_0).$$

Thus $M(\phi \circ u; O, |X|)$ is u.s.c., and the same must be true of s since ϕ is increasing. It is now clear that s is subharmonic in $B(O, R)$. Since it depends only on $|X|$, the proof is completed by appealing to Theorem 2.

4. Convex domains

Let $\Omega \neq \boldsymbol{R}^n$ be a domain (non-empty connected open set) in $\boldsymbol{R}^n$, and let u be the distance function given by $u(X) = -\operatorname{dist}(X, \partial\Omega)$ for $X \in \Omega$. The following elegant result is due to Armitage and Kuran [1].

Theorem 4. *The function u is subharmonic in $\Omega \subset \boldsymbol{R}^2$ if and only if the domain Ω is a convex set.*

Proof. The "if" part of the argument is straightforward. Let L denote an arbitrary straight line $a_L x_1 + b_L x_2 = c_L$ in $\boldsymbol{R}^2 \backslash \Omega$, $(a_L^2 + b_L^2 = 1)$, and let u_L be the signed distance function from L given by

$$u_L = \pm (a_L x_1 + b_L x_2 - c_L),$$

the sign being chosen so that $u_L < 0$ in Ω. Since each u_L is harmonic, $u = \sup_L u_L$ on Ω and u is finite and continuous, it follows from Lemma 1 that u is subharmonic on Ω.
To prove the "only if" part, suppose Ω is not convex. Then it is known [6; Theorem 4.8] that there is a point $Y \in \partial\Omega$ of "strong local concavity". What this means is that, choosing suitable new axes centred at Y, there exists $\varepsilon > 0$ such that

$$\{X = (x_1, x_2): 0 < |X| < 8\varepsilon \text{ and } x_2 \ge 0\} \subset \Omega.$$

Then $d(X) < -x_2$ for $X \in D = B((0, 2\varepsilon), \varepsilon)$ with $x_1 \neq 0$, so

$$\int_D \{d(X) + 2\varepsilon\} \, dx_1 \, dx_2 < \int_D (2\varepsilon - x_2) \, dx_1 \, dx_2 = 0.$$

It now follows that $d(X)$ is not subharmonic in Ω for, if it were, then the subharmonic function $s(X) = d(X) + 2\varepsilon$ would satisfy property (iii a) giving

$$\int_D \{d(X) + 2\varepsilon\}\, dx_1\, dx_2 = \int_0^\varepsilon 2\pi M(s;(0,2\varepsilon),r)\, r\, dr \geq \pi\varepsilon^2 s(0,2\varepsilon) = 0.$$

The same paper gives a counterexample to show that Theorem 4 fails in higher dimensions. For example, when $n = 3$, let Ω be the torus obtained by rotating the disc

$$\{(0,x_2,x_3):(x_2-2)^2 + x_3^2 < 1\}$$

about the x_3-axis. Then it can be shown that u is subharmonic in Ω, yet Ω is clearly not convex. What can be said in higher dimensions is that, if we set $u(X) = \operatorname{dist}(X,\partial\Omega)$ for $X \in \boldsymbol{R}^n \backslash \Omega$, then the function u is subharmonic in the whole of $\boldsymbol{R}^n$ if and only if the domain Ω is a convex set (see [1] for details).

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A note on L'Hôpital's rule

1. Introduction

Recently the classical L'Hôpital's rule, $\lim f/g = \lim f'/g'$, has come again to the centre of interest. Referring to the basic article of Stolz [4], Boas [2] offered a general construction of counterexamples to the rule with non-monotonic g's. He pointed out that not the mere presence of zeros of g', but the infinite number of its sign changes may cause trouble with the rule. Clearly, by the intermediate value property of the derivative, g' can not change sign without having zeros. This is not the case for one-sided derivatives. Starting