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monotonicity theorems. The use of these in the proof of l'Hôpital's rule was made by Lettenmeyer [4]. Since monotonicity theorems are known to hold for Dini derivates, it is clear from our exposition that the right-hand derivatives can be replaced in Theorem 1–2 without affecting their validity by Dini derivates. The following counterexample:

$$f(x) = x + \sin x \cos x, \quad g(x) = f(x)\, e^{\sin x}$$

$\lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)} = 0$ and no limit for $\dfrac{f(x)}{g(x)}$ as $x \to \infty$ was given already in 1879 by O. Stolz [6], who also showed that Theorem 3 (with ordinary rather than one-sided derivatives) can be deduced from Theorem 2. A simple proof based on the Newton-Leibniz formula was given by Boas [2] but one may conjecture that the method was already known to Huntington [3].

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An extension of the isoperimetric inequality on the sphere

We shall consider the n-dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, endowed with the spherical distance function $d(x, y)$ and the (normalized) Lebesgue measure μ. For $x \in S^n$ and $0 \le \theta \le \pi$, the *spherical cap* of centre x and radius θ is $C(x, \theta) = \{y \in S^n : d(x, y) \le \theta\}$. It is well known that if $A \subset S^n$ and $\mu(A) = \mu(C)$ for some spherical cap C, then the diameter of A is at least as large as the diameter of C. This is usually considered to be a variant of the isoperimetric inequality on the sphere S^n; it is, in fact, an immediate consequence of the isoperimetric inequality. Our aim is to extend this inequality and thereby answer a question raised by Paul Erdös [4].

For $k \geq 2$, define the *k-diameter* $d_k(A)$ of a set A in a metric space by

$$d_k(A) = \sup\{\min_{1 \leq i < j \leq k} d(x_i, x_j) : x_1, \ldots, x_k \in A\}.$$

Thus $d_k(A) \leq d$ if and only if A does not contain k points, any two of which are at distance greater than d; in particular, $d_2(A)$ is precisely the diameter of A. We shall show that if $A \subset S^n$ and $0 < \mu(A) = \mu(C)$ for some spherical cap C then $d_k(A) \geq d_k(C)$ for every $k \geq 2$.

The proof we shall give makes use of compression operators and closely follows Benyamini's [2] proof of the classical isoperimetric inequality on the sphere. Benyamini's proof, in turn, was inspired by Baernstein and Taylor [1]. In spirit, the compression operators on the sphere are very close to the compression operators frequently used in the study of set systems in combinatorics (see [3; Chapters 16 and 17 and [4]).

Let A be a subset of S^n and $z \in S^n$. The *compression* $\gamma_z(A)$ *of* A *in the direction of* z *is* defined as follows. For $x \in S^n$, let $x^+ = x - \langle x, z \rangle z + |\langle x, z \rangle| z$ and $x^- = x - \langle x, z \rangle z - |\langle x, z \rangle| z$, where $\langle .,. \rangle$ denotes the inner product in $\mathbb{R}^{n+1}$. Thus the line through x, in the direction of z, meets S^n precisely in x^+ and x^-, where $\langle x^+, z \rangle = -\langle x^-, z \rangle \geq 0$; furthermore, $x^+ = x^-$ if and only if $\langle x, z \rangle = 0$.

The compression operator γ_z pushes the points of $A \cap \{x^+, x^-\}$ towards x^+: if $A \cap \{x^+, x^-\} = \{x^-\}$ then

$$\gamma_z(A) \cap \{x^+, x^-\} = \{x^+\}$$

and if $A \cap \{x^+, x^-\} \neq \{x^-\}$ then

$$\gamma_z(A) \cap \{x^+, x^-\} = A \cap \{x^+, x^-\}.$$

It is trivial that if A is measurable then so is $\gamma_z(A)$ and we have $\mu(\gamma_z(A)) = \mu(A)$; furthermore, if A is closed, so is $\gamma_z(A)$. The compression operators map caps into caps: $\gamma_z(C(x, \theta)) = C(x^+, \theta)$; furthermore, for any two measurable sets A and B,

$$\mu(A \cap B) \leq \mu(\gamma_z(A) \cap \gamma_z(B)). \tag{1}$$

Thus γ_z not only compresses as much of a set A into the hemisphere $\{x^+ : x \in S^n\} = \{x \in S^n : \langle x, z \rangle \geq 0\}$ as possible, but it also compresses sets closer to each other. In this note, the most important property of compression operators is that they do not increase the k-diameter.

Lemma 1. If $A \subset S^n$, $z \in S^n$ and $k \geq 2$ then $d_k(\gamma_z(A)) \leq d_k(A)$.

Proof. It suffices to show that if $d_k(\gamma_z(A)) > d$ then $d_k(A) \geq d$. Let then $d_k(\gamma_z(A)) > d$. Then there is a set $X = \{x_1, \ldots, x_k\} \subset (A)$ with $d(x_i, x_j) \geq d$ for $i \neq j$. We claim that A contains a k-subset X' with minimal distance at least d, so $d_k(A) \geq d$.

In proving this claim we may assume that $x_1, \ldots, x_l$ are the points of $X = \{x_1, \ldots, x_k\}$ that do not belong to A. Then $x_i = x_i^+$ for $1 \leq i \leq l$ and $X' = \{x_1^-, \ldots, x_l^-, x_{l+1}, \ldots, x_k\} \subset A$.

Furthermore, the minimal distance in this k-subset X' of A is at least d. Indeed, for $1 \le i < j \le l$, $d(x_i^-, x_j^-) = d(x_i^+, x_j^+) \ge d$ since $x_i^+, x_j^+ \in X$, and for $l+1 \le i < j \le k$ we have $d(x_i, x_j) \ge d$ since $x_i, x_j \in X$. Let now $1 \le i \le l$ and $l+1 \le j \le k$. If $x_j = x_j^-$ then $d(x_i^-, x_j^-) = d(x_i^-, x_j^-) = d(x_i^+, x_j^+) \ge d$ since $x_i^+, x_j^+ \in X$. Finally, if $x_j = x_j^+$ then $d(x_i^-, x_j) = d(x_i^-, x_j^+) \ge d(x_i^+, x_j^+) \ge d$ since $x_i^+, x_j^+ \in X$. $\square$

Loosely speaking, our aim is to show that if A is a closed subset of S^n then A can gradually be transformed into a spherical cap of measure at least $\mu(A)$ and k-diameter at most $d_k(A)$. Lemma 1 tells us that A can transformed into $\gamma_z(A)$ for every $z \in S^n$. The next lemma, which is essentially trivial, shows that we can take limits in the Hausdorff metric: the k-diameter is continuous in this metric and, in fact, every Borel measure on S^n is upper semi-continuous. Let H be the metric space of closed non-empty subsets of S^n with the Hausdorff metric $d(A, B) = \sup\{d(a, B), d(b, A) : a \in A, b \in B\}$. Since S^n is compact, H is also a compact metric space.

Lemma 2. Let v be a Borel measure on S^n and let $A, A_1, A_2, \ldots \in H$, $A_s \to A$. Then

$$v(A) \ge \lim_{s \to \infty} v(A_s) \quad \text{and} \quad d_k(A) = \lim_{s \to \infty} d_k(A_s).$$

Proof. (i) Given $\varepsilon > 0$, let $\delta > 0$ be such that $v(A_\delta) < v(A) + \varepsilon$, where $A_\delta = \{x \in S^n : d(x, A) < \delta\}$. If s is large enough then $A_s \subset A_\delta$ so $v(A_s) < v(A) + \varepsilon$, showing that v is upper semi-continuous.
(ii) Suppose $d(A, B) < \delta$ where $A, B \in H$, and $x_1, \ldots, x_k \in A$. Then for each x_i there is a $y_i \in B$ such that $d(x_i, y_i) < \delta$. Clearly $d(y_i, y_j) > d(x_i, x_j) - 2\delta$ so $d_k(B) \ge d_k(A) - 2\delta$. Interchanging A and B we see that $d_k(A) \ge d_k(B) - 2\delta$. Hence, given $\varepsilon > 0$, if s is large enough to guarantee that $d(A_s, A) < \varepsilon/2$ then we have $|d_k(A_s) - d_k(A)| \le \varepsilon$. $\square$

We are ready to prove the main result of this note. As usual, we shall write μ^* for the outer measure defined by μ.

Theorem 3. Let A be a non-empty subset of S^n and let C be a cap of measure $\mu^*(A)$. Then $d_k(A) \ge d_k(C)$ for every $k \ge 2$.

Proof. The assertion is trivial if $\mu^*(A) = 0$ or $\mu^*(A) = \mu(S^n)$. Furthermore, since $d_k(A) = d_k(\bar{A})$, we may assume that A is a closed set of measure m, $0 < m < \mu(S^n)$.
Let K be the minimal closed subset of H containing A and closed under γ_z for every $z \in S^n$. By Lemmas 1 and 2, every set in K has measure at least m and k-diameter at most $d_k(A)$. For a Borel subset of M of S^n, define $v(M) = \mu(M \cap C)$, where C is our spherical cap of measure m. Then v is a Borel measure on S^n; by Lemma 2, this measure v is upper semi-continuous so its supremum on K is attained on some set $M \in K$. To complete the proof, we shall show that M contains the cap C.
Suppose that this is not the case. Then there is a cap $= C(x, \theta)$, $\theta > 0$, such that $D \subset C \backslash M$. Since $\mu(M) \ge \mu(C)$, this implies that $\mu(M \backslash C) > 0$ so there is a cap $E = C(y, \mu)$, $0 < \mu \le \theta$, such that $E \cap C = 0$ and $\mu(M \cap E) > 0$. By replacing θ by μ, we may assume that $\mu = \theta$.

Let $z = (x - y)/\|x - y\|$. Then $\gamma_z(E) = D$, $\gamma_z(C) = C$ and $\gamma_z(C \backslash D) = C \backslash D$. Hence, by (1),

$$\begin{aligned}\mu(\gamma_z(M) \cap C) &= \mu(\gamma_z(M) \cap (C \backslash D)) + \mu(\gamma_z(M) \cap D) \\ &\geq \mu(M \cap (C \backslash D)) + \mu(M \cap E) = \mu(M \cap C) + \mu(M \cap E) > \mu(M \cap C).\end{aligned}$$

Since $\gamma_z(M) \in K$, this contradicts the choice of M, so the proof is complete. □

Let us remark that a slight variant of the proof above gives the following assertion. Let K be a non-empty closed subset of H which is also closed under the operators γ_z, i.e. which is such that $\gamma_z(A) \in K$ for all $A \in K$ and $z \in S^n$. Then K contains all caps of measure $m = \sup\{\mu(A): A \in K\}$.

Also, it is easily seen that the proof above implies various extensions of Theorem 3. For example, given finite sets $X, Y \subset S^n$ with $|X| = |Y|$, let us write $X \leq Y$ if for every $d > 0$, the number of pairs in X at distance at least d is not more than the number of pairs in Y at distance at least d. Furthermore, for sets $A, B \subset S^n$, let us write $A \leq B$ for the assertion that for every finite set $X \subset A$ there is a finite set $Y \subset B$ with $|Y| = |X|$ and $X \leq Y$. Then the following assertion holds. Let A be a non-empty closed subset of S^n and let C be a cap of measure $\mu(A)$. Then $C \leq A$.

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Winch curves

A taut rope connects a point in the origin of a rectangular coordinate system with a point in $R(a, 0)$. If the latter starts moving along the line $x = a$, it will trail the point in the origin. For each point P of the curve that is created in this way we have $PQ = a$, where Q is the intersection of the tangent to the curve in P with the line $x = a$. This curve, known as the tractrix, is represented by an equation that can be found as follows.
In the rectangular triangle PSQ (see fig. 1) we have

$$PQ = a, \quad PS = a - x, \quad SQ = (a - x)\,dy/dx.$$