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monotonicity theorems. The use of these in the proof of l'Hôpital's rule was made by Lettenmeyer [4]. Since monotonicity theorems are known to hold for Dini derivatives, it is clear from our exposition that the right-hand derivatives can be replaced in Theorem 1–2 without affecting their validity by Dini derivatives. The following counterexample:

$$f(x) = x + \sin x \cos x, \quad g(x) = f(x) e^{\sin x}$$

$\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} = 0$  and no limit for  $\frac{f(x)}{g(x)}$  as  $x \rightarrow \infty$  was given already in 1879 by O. Stolz [6],

who also showed that Theorem 3 (with ordinary rather than one-sided derivatives) can be deduced from Theorem 2. A simple proof based on the Newton-Leibniz formula was given by Boas [2] but one may conjecture that the method was already known to Huntington [3].

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## An extension of the isoperimetric inequality on the sphere

We shall consider the  $n$ -dimensional sphere  $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ , endowed with the spherical distance function  $d(x, y)$  and the (normalized) Lebesgue measure  $\mu$ . For  $x \in S^n$  and  $0 \leq \theta \leq \pi$ , the *spherical cap* of centre  $x$  and radius  $\theta$  is  $C(x, \theta) = \{y \in S^n : d(x, y) \leq \theta\}$ . It is well known that if  $A \subset S^n$  and  $\mu(A) = \mu(C)$  for some spherical cap  $C$ , then the diameter of  $A$  is at least as large as the diameter of  $C$ . This is usually considered to be a variant of the isoperimetric inequality on the sphere  $S^n$ ; it is, in fact, an immediate consequence of the isoperimetric inequality. Our aim is to extend this inequality and thereby answer a question raised by Paul Erdős [4].

For  $k \geq 2$ , define the  $k$ -diameter  $d_k(A)$  of a set  $A$  in a metric space by

$$d_k(A) = \sup \left\{ \min_{1 \leq i < j \leq k} d(x_i, x_j) : x_1, \dots, x_k \in A \right\}.$$

Thus  $d_k(A) \leq d$  if and only if  $A$  does not contain  $k$  points, any two of which are at distance greater than  $d$ ; in particular,  $d_2(A)$  is precisely the diameter of  $A$ . We shall show that if  $A \subset S^n$  and  $0 < \mu(A) = \mu(C)$  for some spherical cap  $C$  then  $d_k(A) \geq d_k(C)$  for every  $k \geq 2$ .

The proof we shall give makes use of compression operators and closely follows Benyamini's [2] proof of the classical isoperimetric inequality on the sphere. Benyamini's proof, in turn, was inspired by Baernstein and Taylor [1]. In spirit, the compression operators on the sphere are very close to the compression operators frequently used in the study of set systems in combinatorics (see [3; Chapters 16 and 17 and [4]).

Let  $A$  be a subset of  $S^n$  and  $z \in S^n$ . The *compression*  $\gamma_z(A)$  of  $A$  in the direction of  $z$  is defined as follows. For  $x \in S^n$ , let  $x^+ = x - \langle x, z \rangle z + |\langle x, z \rangle| z$  and  $x^- = x - \langle x, z \rangle z - |\langle x, z \rangle| z$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{R}^{n+1}$ . Thus the line through  $x$ , in the direction of  $z$ , meets  $S^n$  precisely in  $x^+$  and  $x^-$ , where  $\langle x^+, z \rangle = -\langle x^-, z \rangle \geq 0$ ; furthermore,  $x^+ = x^-$  if and only if  $\langle x, z \rangle = 0$ .

The compression operator  $\gamma_z$  pushes the points of  $A \cap \{x^+, x^-\}$  towards  $x^+$ : if  $A \cap \{x^+, x^-\} = \{x^-\}$  then

$$\gamma_z(A) \cap \{x^+, x^-\} = \{x^+\}$$

and if  $A \cap \{x^+, x^-\} \neq \{x^-\}$  then

$$\gamma_z(A) \cap \{x^+, x^-\} = A \cap \{x^+, x^-\}.$$

It is trivial that if  $A$  is measurable then so is  $\gamma_z(A)$  and we have  $\mu(\gamma_z(A)) = \mu(A)$ ; furthermore, if  $A$  is closed, so is  $\gamma_z(A)$ . The compression operators map caps into caps:  $\gamma_z(C(x, \theta)) = C(x^+, \theta)$ ; furthermore, for any two measurable sets  $A$  and  $B$ ,

$$\mu(A \cap B) \leq \mu(\gamma_z(A) \cap \gamma_z(B)). \quad (1)$$

Thus  $\gamma_z$  not only compresses as much of a set  $A$  into the hemisphere  $\{x^+ : x \in S^n\} = \{x \in S^n : \langle x, z \rangle \geq 0\}$  as possible, but it also compresses sets closer to each other. In this note, the most important property of compression operators is that they do not increase the  $k$ -diameter.

**Lemma 1.** If  $A \subset S^n$ ,  $z \in S^n$  and  $k \geq 2$  then  $d_k(\gamma_z(A)) \leq d_k(A)$ .

*Proof.* It suffices to show that if  $d_k(\gamma_z(A)) > d$  then  $d_k(A) \geq d$ . Let then  $d_k(\gamma_z(A)) > d$ . Then there is a set  $X = \{x_1, \dots, x_k\} \subset \gamma_z(A)$  with  $d(x_i, x_j) \geq d$  for  $i \neq j$ . We claim that  $A$  contains a  $k$ -subset  $X'$  with minimal distance at least  $d$ , so  $d_k(A) \geq d$ .

In proving this claim we may assume that  $x_1, \dots, x_l$  are the points of  $X = \{x_1, \dots, x_k\}$  that do not belong to  $A$ . Then  $x_i = x_i^+$  for  $1 \leq i \leq l$  and  $X' = \{x_1^-, \dots, x_l^-, x_{l+1}, \dots, x_k\} \subset A$ .

Furthermore, the minimal distance in this  $k$ -subset  $X'$  of  $A$  is at least  $d$ . Indeed, for  $1 \leq i < j \leq l$ ,  $d(x_i^-, x_j^-) = d(x_i^+, x_j^+) \geq d$  since  $x_i^+, x_j^+ \in X$ , and for  $l + 1 \leq i < j \leq k$  we have  $d(x_i, x_j) \geq d$  since  $x_i, x_j \in X$ . Let now  $1 \leq i \leq l$  and  $l + 1 \leq j \leq k$ . If  $x_j = x_j^-$  then  $d(x_i^-, x_j^-) = d(x_i^-, x_j^-) = d(x_i^+, x_j^+) \geq d$  since  $x_i^+, x_j^+ \in X$ . Finally, if  $x_j = x_j^+$  then  $d(x_i^-, x_j) = d(x_i^-, x_j^+) \geq d(x_i^+, x_j^+) \geq d$  since  $x_i^+, x_j^+ \in X$ .  $\square$

Loosely speaking, our aim is to show that if  $A$  is a closed subset of  $S^n$  then  $A$  can gradually be transformed into a spherical cap of measure at least  $\mu(A)$  and  $k$ -diameter at most  $d_k(A)$ . Lemma 1 tells us that  $A$  can be transformed into  $\gamma_z(A)$  for every  $z \in S^n$ . The next lemma, which is essentially trivial, shows that we can take limits in the Hausdorff metric: the  $k$ -diameter is continuous in this metric and, in fact, every Borel measure on  $S^n$  is upper semi-continuous. Let  $H$  be the metric space of closed non-empty subsets of  $S^n$  with the Hausdorff metric  $d(A, B) = \sup \{d(a, B), d(b, A) : a \in A, b \in B\}$ . Since  $S^n$  is compact,  $H$  is also a compact metric space.

**Lemma 2.** Let  $\nu$  be a Borel measure on  $S^n$  and let  $A, A_1, A_2, \dots \in H, A_s \rightarrow A$ . Then

$$\nu(A) \geq \lim_{s \rightarrow \infty} \nu(A_s) \quad \text{and} \quad d_k(A) = \lim_{s \rightarrow \infty} d_k(A_s).$$

*Proof.* (i) Given  $\varepsilon > 0$ , let  $\delta > 0$  be such that  $\nu(A_\delta) < \nu(A) + \varepsilon$ , where  $A_\delta = \{x \in S^n : d(x, A) < \delta\}$ . If  $s$  is large enough then  $A_s \subset A_\delta$  so  $\nu(A_s) < \nu(A) + \varepsilon$ , showing that  $\nu$  is upper semi-continuous.

(ii) Suppose  $d(A, B) < \delta$  where  $A, B \in H$ , and  $x_1, \dots, x_k \in A$ . Then for each  $x_i$  there is a  $y_i \in B$  such that  $d(x_i, y_i) < \delta$ . Clearly  $d(y_i, y_j) > d(x_i, x_j) - 2\delta$  so  $d_k(B) \geq d_k(A) - 2\delta$ . Interchanging  $A$  and  $B$  we see that  $d_k(A) \geq d_k(B) - 2\delta$ . Hence, given  $\varepsilon > 0$ , if  $s$  is large enough to guarantee that  $d(A_s, A) < \varepsilon/2$  then we have  $|d_k(A_s) - d_k(A)| \leq \varepsilon$ .  $\square$

We are ready to prove the main result of this note. As usual, we shall write  $\mu^*$  for the outer measure defined by  $\mu$ .

**Theorem 3.** Let  $A$  be a non-empty subset of  $S^n$  and let  $C$  be a cap of measure  $\mu^*(A)$ . Then  $d_k(A) \geq d_k(C)$  for every  $k \geq 2$ .

*Proof.* The assertion is trivial if  $\mu^*(A) = 0$  or  $\mu^*(A) = \mu(S^n)$ . Furthermore, since  $d_k(A) = d_k(\bar{A})$ , we may assume that  $A$  is a closed set of measure  $m, 0 < m < \mu(S^n)$ .

Let  $K$  be the minimal closed subset of  $H$  containing  $A$  and closed under  $\gamma_z$  for every  $z \in S^n$ . By Lemmas 1 and 2, every set in  $K$  has measure at least  $m$  and  $k$ -diameter at most  $d_k(A)$ . For a Borel subset  $M$  of  $S^n$ , define  $\nu(M) = \mu(M \cap C)$ , where  $C$  is our spherical cap of measure  $m$ . Then  $\nu$  is a Borel measure on  $S^n$ ; by Lemma 2, this measure  $\nu$  is upper semi-continuous so its supremum on  $K$  is attained on some set  $M \in K$ . To complete the proof, we shall show that  $M$  contains the cap  $C$ .

Suppose that this is not the case. Then there is a cap  $= C(x, \theta), \theta > 0$ , such that  $D \subset C \setminus M$ . Since  $\mu(M) \geq \mu(C)$ , this implies that  $\mu(M \setminus C) > 0$  so there is a cap  $E = C(y, \mu), 0 < \mu \leq \theta$ , such that  $E \cap C = \emptyset$  and  $\mu(M \cap E) > 0$ . By replacing  $\theta$  by  $\mu$ , we may assume that  $\mu = \theta$ .

Let  $z = (x - y)/\|x - y\|$ . Then  $\gamma_z(E) = D$ ,  $\gamma_z(C) = C$  and  $\gamma_z(C \setminus D) = C \setminus D$ . Hence, by (1),

$$\begin{aligned} \mu(\gamma_z(M) \cap C) &= \mu(\gamma_z(M) \cap (C \setminus D)) + \mu(\gamma_z(M) \cap D) \\ &\geq \mu(M \cap (C \setminus D)) + \mu(M \cap D) = \mu(M \cap C) + \mu(M \cap D) > \mu(M \cap C). \end{aligned}$$

Since  $\gamma_z(M) \in K$ , this contradicts the choice of  $M$ , so the proof is complete.  $\square$

Let us remark that a slight variant of the proof above gives the following assertion. Let  $K$  be a non-empty closed subset of  $H$  which is also closed under the operators  $\gamma_z$ , i.e. which is such that  $\gamma_z(A) \in K$  for all  $A \in K$  and  $z \in S^n$ . Then  $K$  contains all caps of measure  $m = \sup \{ \mu(A) : A \in K \}$ .

Also, it is easily seen that the proof above implies various extensions of Theorem 3. For example, given finite sets  $X, Y \subset S^n$  with  $|X| = |Y|$ , let us write  $X \leq Y$  if for every  $d > 0$ , the number of pairs in  $X$  at distance at least  $d$  is not more than the number of pairs in  $Y$  at distance at least  $d$ . Furthermore, for sets  $A, B \subset S^n$ , let us write  $A \leq B$  for the assertion that for every finite set  $X \subset A$  there is a finite set  $Y \subset B$  with  $|Y| = |X|$  and  $X \leq Y$ . Then the following assertion holds. Let  $A$  be a non-empty closed subset of  $S^n$  and let  $C$  be a cap of measure  $\mu(A)$ . Then  $C \leq A$ .

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## Winch curves

A taut rope connects a point in the origin of a rectangular coordinate system with a point in  $R(a, 0)$ . If the latter starts moving along the line  $x = a$ , it will trail the point in the origin. For each point  $P$  of the curve that is created in this way we have  $PQ = a$ , where  $Q$  is the intersection of the tangent to the curve in  $P$  with the line  $x = a$ . This curve, known as the tractrix, is represented by an equation that can be found as follows.

In the rectangular triangle  $PSQ$  (see fig. 1) we have

$$PQ = a, \quad PS = a - x, \quad SQ = (a - x) dy/dx.$$