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Kleine Mitteilungen

An explicit formula about the convex hull of random points

Denote by $V_n^{(d)}$ the expected volume of the convex hull of n points chosen independently according to a given probability measure μ in Euclidean d-space E^d. For $d = 2, 3$ and μ the uniform distribution on a convex body in E^d, Affentranger [1], [2] has shown that

$$V_{d+2m}^{(d)} = \sum_{k=1}^{m} \gamma_k \binom{d+2m}{2k-1} V_{d+2m-2k+1}^{(d)} \quad (m = 1, 2, \ldots), \tag{1}$$

where the γ_k can be obtained recursively from $\gamma_1 = \frac{1}{2}$, $2\gamma_k = 1 - \sum_{i=1}^{k-1} \binom{2k-1}{2i-1} \gamma_i \, (k \geq 2)$.

Recently, Buchta [3] has extended this result to arbitrary dimensions d and to arbitrary probability measures μ on E^d. The key point in [3] is the existence of a moment functional $\mathcal{M}$ such that

$$V_{d+1+n}^{(d)} = \binom{d+1+n}{d+1} \mathcal{M}(x^n + (1-x)^n). \tag{2}$$

(See [4] for the definition of moment functionals.)
In this note we show that in formula (1) the γ_k can be expressed explicitly by

$$\gamma_k = (2^{2k} - 1)\frac{B_{2k}}{k} \quad (k = 1, 2, \ldots). \tag{3}$$

Here the B_n are the *Bernoulli numbers* (see e.g. [5], section 1.13), defined by the generating series $z/(e^z - 1) = \sum_{n=0}^{\infty} B_n z^n/n!$. In our proof of formula (1) we can avoid the elimination process used in [1].

The *Euler polynomials* $E_n(x)$ (see e.g. [5], section 1.14) are defined by the generating series

$$\frac{2e^{xz}}{e^z+1} = \sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} \quad (|z| < \pi). \tag{4}$$

Usually, the Euler polynomials are expanded in powers of $x - \frac{1}{2}$. We need the less known expansion in powers of x. Thus put $\tilde{B}_n = E_n(0)$. Then the numbers $\tilde{B}_n$ are defined by the generating series

$$\frac{2}{e^z+1} = \sum_{n=0}^{\infty} \tilde{B}_n \frac{z^n}{n!} \quad (|z| < \pi). \tag{5}$$

Differentiation of $2/(e^z+1)$ gives the even function $-2e^z/(e^z+1)^2$; hence $\tilde{B}_{2n} = 0$ for $n \geq 1$. Since

$$\frac{2z}{e^z+1} = \frac{2z}{e^z-1} - \frac{4z}{e^{2z}-1},$$

we can also express $\tilde{B}_n$ in terms of the Bernoulli number B_{n+1}:

$$\tilde{B}_n = -(2^{n+1}-1)\frac{2B_{n+1}}{n+1}. \tag{6}$$

We read off from (4) that $E_0(x) = 1$, $E_n'(x) = nE_{n-1}(x)$. This leads to

$$E_n(x) = x^n + \sum_{k=1}^{n} \binom{n}{k} \tilde{B}_k x^{n-k}. \tag{7}$$

Next, we see from $2e^{(1-x)z}/(e^z+1) = 2e^{-xz}/(e^{-z}+1)$ that $E_{2m-1}(x) + E_{2m-1}(1-x) = 0$. Now we obtain the key expansion by substituting this in equation (7) and by using $\tilde{B}_{2k} = 0$:

$$x^{2m-1} + (1-x)^{2m-1} = -\sum_{k=1}^{m} \binom{2m-1}{2k-1} \tilde{B}_{2k-1}(x^{2m-2k} + (1-x)^{2m-2k}). \tag{8}$$

Finally we use expression (2) for $V_{d+2m}^{(d)}$ and substitute it into equation (8). This gives

$$\begin{aligned} V_{d+2m}^{(d)} &= -\binom{d+2m}{d+1} \sum_{k=1}^{m} \binom{2m-1}{2k-1} \tilde{B}_{2k-1} \binom{d+2m-2k+1}{d+1}^{-1} V_{d+2m-2k+1}^{(d)} \\ &= -\sum_{k=1}^{m} \binom{d+2m}{2k-1} \tilde{B}_{2k-1} V_{d+2m-2k+1}^{(d)}. \end{aligned}$$

In view of (6), this proves (1) and (3).

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Aufgaben

Aufgabe 989. Es seien $\alpha_1, \alpha_2, \alpha_3$ die Winkel eines ebenen Dreiecks und $\alpha_1 \leqq \alpha_2, \alpha_3$. Dann gilt die Ungleichung

$$\alpha_1 \leqq 2 \operatorname{arccot}(\sqrt{3} + \sum (1/\alpha_i) - 9/\pi)$$

mit Gleichheit genau im gleichseitigen Fall. Dies ist zu beweisen.

V. D. Mascioni, Origlio

Lösung. O.B.d.A. darf noch $\alpha_2 \leqq \alpha_3$ vorausgesetzt werden, was mit $\alpha_2 \leqq (\pi - \alpha_1)/2$ äquivalent ist. Da die behauptete Ungleichung mit

$$(*) \quad \cot(\alpha_1/2) - 1/\alpha_1 + 9/\pi - \sqrt{3} \geqq 1/\alpha_2 + 1/(\pi - \alpha_1 - \alpha_2)$$

für $\alpha_1 \leqq \alpha_2 \leqq (\pi - \alpha_1)/2$ gleichbedeutend ist, wobei $0 < \alpha_1 \leqq \pi/3$ zu gelten hat, genügt es festzustellen, dass die rechte Seite von (*), betrachtet als Funktion von α_2, im Intervall $[\alpha_1, (\pi - \alpha_1)/2]$ monoton fällt. Daher ist (*) richtig, sobald

$$(**) \quad F(\alpha_1) := \cot(\alpha_1/2) - 2/\alpha_1 - 1/(\pi - 2\alpha_1) \geqq \sqrt{3} - 9/\pi$$

für alle $\alpha_1 \in]0, \pi/3]$ bewiesen ist. Aus $\sin^2 y \leqq y^2$ für alle reellen y folgt

$$F'(x) = -1/(2\sin^2 \tfrac{x}{2}) + 2/x^2 - 2/(\pi - 2x)^2 < -2/(\pi - 2x)^2,$$

weshalb F in $]0, \pi/3]$ streng monoton fällt. Damit ist $F(\alpha_1) \geqq F(\pi/3) = \sqrt{3} - 9/\pi$ für $0 < \alpha_1 \leqq \pi/3$ mit Gleichheit genau für $\alpha_1 = \pi/3$, d.h. im Fall der Gleichseitigkeit.

P. Bundschuh, Köln, BRD

Weitere Lösungen sandten W. Janous (Innsbruck, A), L. Kuipers (Sierre), Kee-wai Lau (Hong-Kong), B. Ruh (Zürich), P. Weisenhorn (Achern, BRD).
Eine Lösung war fehlerhaft.