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## A Desarguesian dual for Nagel's middlespoint

1. In a paper published in 1836, C. H. Nagel [4] defines the “middlespoint” (Mittenpunkt) of a given triangle  $ABC$  in the following manner:

*Let  $S_A, S_B, S_C$  be the midpoints of  $BC, CA, AB$  respectively and  $I_a, I_b, I_c$  the centres of the excircles, then the lines  $S_A I_a, S_B I_b, S_C I_c$  concur at  $M$ , the middlespoint of  $ABC$ ,*

see also [1]. The name is probably derived from the fact that the point is constructed using “middles”, namely, *centres* of circles and *midpoints* of line segments. In this paper, we derive a dual (line) for this remarkable, but seemingly little known, point and show how this new line relates to some known geometry of the triangle.

2. Desargues's two-triangle theorem in the plane states that if triangles  $ABC$  and  $A_1 B_1 C_1$  are perspective from a point  $L$ , they are perspective from a line  $l$ , i.e. if  $AA_1 \cap BB_1 \cap CC_1 = L$ , then  $(AB \cap A_1 B_1) \cup (BC \cap B_1 C_1) \cup (CA \cap C_1 A_1) = l$ . Clearly, the converse of this theorem is also its dual, hence, for purposes of this paper, we refer to  $L$  and  $l$  as “Desarguesian duals”. Also, we will have occasion to make reference to the special case when the triangle  $A_1 B_1 C_1$  is inscribed in  $ABC$ , i.e.  $A_1$  is on  $BC$ , etc. In this instance,  $L$  is called the trilinear pole of  $l$  and, dually,  $l$  is the trilinear polar of  $L$ , see [2].

3. In order to facilitate the arguments, we shall use a system of homogeneous coordinates called “trilinear” or “normal”. In order to avoid a possible confusion with trilinear poles and polars, we shall use the term “normal” throughout. In this system, the coordinates  $(x, y, z)$  of a point  $L$  in the plane of  $ABC$  are proportional to the signed distances  $d_a, d_b, d_c$  of  $L$  from the sides of the triangle of reference  $ABC$ , where, obviously,  $A = (1, 0, 0)$ ,  $B = (0, 1, 0)$ ,  $C = (0, 0, 1)$ . The distance  $d_a$ , for example, is positive if  $L$  and the unit point  $I = (1, 1, 1)$ , the incentre, are on the same side of  $a = BC$  and negative otherwise. For instance, the excentre  $I_A$  of the excircle opposite vertex  $A$  has coordinates  $(-1, 1, 1)$ , or its projective equivalent,  $(1, -1, -1)$ .

We now derive the normal line coordinates of  $l$ , the trilinear polar of  $L$ , that is, if a line  $t$  has equation  $ux + vy + wz = 0$  then  $t = [u, v, w]$  is its normal representation. Also, for the remainder of the paper, we shall denote  $A_1$  by  $A_L$ , etc., thus  $A_L = (0, y, z)$ ,  $B_L = (x, 0, z)$ ,  $C_L = (x, y, 0)$  and, consequently,  $A_L B_L = [yz, xz, -xy] = \left[ \frac{1}{x}, \frac{1}{y}, -\frac{1}{z} \right]$ ,  $xyz \neq 0$ . Now  $C_L'^{[1*]} = AB \cap A_L B_L = (x, -y, 0)$  and, similarly  $B_L' = (-x, 0, z)$ ,  $A_L' = (0, y, -z)$ ; hence the coordinates of  $l = A_L' B_L' C_L'$  readily follow. Since this result does not seem to appear in the available literature, we state it as a proposition.

**Proposition 1.** If a point  $L = (x, y, z)$  is in the plane of, but not incident with, a given triangle of reference  $ABC$  then,  $l$ , its trilinear polar, has coordinates  $\left[ \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right]$ .

4. The coordinates of the middlespoint  $M$  are also readily obtained. The coordinates of the centroid  $S$  are easily seen to be of the form  $\left( \frac{1}{\sin \alpha}, \frac{1}{\sin \beta}, \frac{1}{\sin \gamma} \right)$ , where  $\alpha, \beta, \gamma$  are

the measures of the vertex angles at  $A, B, C$  respectively, however, since  $a = 2R \sin \alpha$ , where  $R$  is the circumradius of  $ABC$ , it is more convenient to write  $S = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ .

Now  $I_a S_A = [(b-c), b, -c]$ ,  $I_b S_B = [-a, (c-a), c]$ ,  $I_c S_C = [a, -b, (a-b)]$ , consequently,  $M = (b+c-a, c+a-b, a+b-c)$ . Again, we state this result in the form of a proposition.

**Proposition 2.** The normal coordinates of the midpoint  $M$  of a given triangle  $ABC$  are given by  $M = (s-a, s-b, s-c)$ , where  $s = \frac{a+b+c}{2}$ .

5. Since the triangles  $I_A I_B I_C$  and  $S_A S_B S_C$  are perspective from the midpoint  $M$ , they are, by Desargues's theorem, perspective from a line  $m$ , the Desarguesian dual of  $M$ , which we shall call the "middlesline" (mittenlinie) of  $ABC$ . Following the procedures above, it is an elementary exercise to show that the coordinates of  $m$  are  $[a(s-a), b(s-b), c(s-c)]$ , the details of which we leave as an exercise for the reader. We now state and prove a related result.

**Proposition 3.** The middlesline is the trilinear polar of the Gergonne point  $G$  of the given triangle  $ABC$ .

Proof: Since  $G_A, G_B, G_C$  are the points of contact of the incircle with the sides of  $ABC$ ,  $BG_A = s-b$  and  $G_A C = s-c$ , hence  $G_A = (0, c(s-c), b(s-b))$  with similiar expressions for  $G_B, G_C$ . It now follows that the coordinates of  $G$  are  $\left(\frac{1}{a(s-a)}, \frac{1}{b(s-b)}, \frac{1}{c(s-c)}\right)$  and by proposition 1, its trilinear polar is  $[a(s-a), b(s-b), c(s-c)] = m$  as claimed.

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#### NOTE

[1\*] The "prime" notation is particularly useful here since  $C_L$  and  $C'_L$  are related by the harmonic conjugacy involution, see [2].