

**Zeitschrift:** Elemente der Mathematik  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 44 (1989)  
**Heft:** 3

**Artikel:** A Desarguesian dual for Nagel's middlespoint  
**Autor:** Eddy, Roland H.  
**DOI:** <https://doi.org/10.5169/seals-41613>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 07.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## A Desarguesian dual for Nagel's middlespoint

- 1.** In a paper published in 1836, C. H. Nagel [4] defines the “middlespoint”(Mittenpunkt) of a given triangle  $ABC$  in the following manner:

*Let  $S_A, S_B, S_C$  be the midpoints of  $BC, CA, AB$  respectively and  $I_a, I_b, I_c$  the centres of the excircles, then the lines  $S_A I_a, S_B I_b, S_C I_c$  concur at  $M$ , the middlespoint of  $ABC$ ,*

see also [1]. The name is probably derived from the fact that the point is constructed using “middles”, namely, *centres* of circles and *midpoints* of line segments. In this paper, we derive a dual (line) for this remarkable, but seemingly little known, point and show how this new line relates to some known geometry of the triangle.

- 2.** Desargues's two-triangle theorem in the plane states that if triangles  $ABC$  and  $A_1B_1C_1$  are perspective from a point  $L$ , they are perspective from a line  $l$ , i.e. if  $AA_1 \cap BB_1 \cap CC_1 = L$ , then  $(AB \cap A_1B_1) \cup (BC \cap B_1C_1) \cup (CA \cap C_1A_1) = l$ . Clearly, the converse of this theorem is also its dual, hence, for purposes of this paper, we refer to  $L$  and  $l$  as “Desarguesian duals”. Also, we will have occasion to make reference to the special case when the triangle  $A_1B_1C_1$  is inscribed in  $ABC$ , i.e.  $A_1$  is on  $BC$ , etc. In this instance,  $L$  is called the trilinear pole of  $l$  and, dually,  $l$  is the trilinear polar of  $L$ , see [2].

- 3.** In order to facilitate the arguments, we shall use a system of homogeneous coordinates called “trilinear” or “normal”. In order to avoid a possible confusion with trilinear poles and polars, we shall use the term “normal” throughout. In this system, the coordinates  $(x, y, z)$  of a point  $L$  in the plane of  $ABC$  are proportional to the signed distances  $d_a, d_b, d_c$  of  $L$  from the sides of the triangle of reference  $ABC$ , where, obviously,  $A = (1, 0, 0)$ ,  $B = (0, 1, 0)$ ,  $C = (0, 0, 1)$ . The distance  $d_a$ , for example, is positive if  $L$  and the unit point  $I = (1, 1, 1)$ , the incentre, are on the same side of  $a = BC$  and negative otherwise. For instance, the excentre  $I_A$  of the excircle opposite vertex  $A$  has coordinates  $(-1, 1, 1)$ , or its projective equivalent,  $(1, -1, -1)$ .

We now derive the normal line coordinates of  $l$ , the trilinear polar of  $L$ , that is, if a line  $t$  has equation  $ux + vy + wz = 0$  then  $t = [u, v, w]$  is its normal representation. Also, for the remainder of the paper, we shall denote  $A_1$  by  $A_L$ , etc., thus  $A_L = (0, y, z)$ ,  $B_L = (x, 0, z)$ ,  $C_L = (x, y, 0)$  and, consequently,  $A_L B_L = [yz, xz, -xy] = \left[ \frac{1}{x}, \frac{1}{y}, -\frac{1}{z} \right]$ ,  $xyz \neq 0$ . Now  $C_L^{[1^*]} = AB \cap A_L B_L = (x, -y, 0)$  and, similarly  $B'_L = (-x, 0, z)$ ,  $A'_L = (0, y, -z)$ ; hence the coordinates of  $l = A'_L B'_L C'_L$  readily follow. Since this result does not seem to appear in the available literature, we state it as a proposition.

**Proposition 1.** If a point  $L = (x, y, z)$  is in the plane of, but not incident with, a given triangle of reference  $ABC$  then,  $l$ , its trilinear polar, has coordinates  $\left[ \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right]$ .

- 4.** The coordinates of the middlespoint  $M$  are also readily obtained. The coordinates of the centroid  $S$  are easily seen to be of the form  $\left( \frac{1}{\sin \alpha}, \frac{1}{\sin \beta}, \frac{1}{\sin \gamma} \right)$ , where  $\alpha, \beta, \gamma$  are

the measures of the vertex angles at  $A, B, C$  respectively, however, since  $a = 2R \sin \alpha$ , where  $R$  is the circumradius of  $ABC$ , it is more convenient to write  $S = \left( \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right)$ .

Now  $I_a S_A = [(b-c), b, -c]$ ,  $I_b S_B = [-a, (c-a), c]$ ,  $I_c S_C = [a, -b, (a-b)]$ , consequently,  $M = (b+c-a, c+a-b, a+b-c)$ . Again, we state this result in the form of a proposition.

**Proposition 2.** The normal coordinates of the middlespoint  $M$  of a given triangle  $ABC$  are given by  $M = (s-a, s-b, s-c)$ , where  $s = \frac{a+b+c}{2}$ .

5. Since the triangles  $I_A I_B I_C$  and  $S_A S_B S_C$  are perspective from the middlespoint  $M$ , they are, by Desargues's theorem, perspective from a line  $m$ , the Desarguesian dual of  $M$ , which we shall call the "middlesline" (mittelinie) of  $ABC$ . Following the procedures above, it is an elementary exercise to show that the coordinates of  $m$  are  $[a(s-a), b(s-b), c(s-c)]$ , the details of which we leave as an exercise for the reader. We now state and prove a related result.

**Proposition 3.** The middlesline is the trilinear polar of the Gergonne point  $G$  of the given triangle  $ABC$ .

Proof: Since  $G_A, G_B, G_C$  are the points of contact of the incircle with the sides of  $ABC$ ,  $BG_A = s-b$  and  $G_A C = s-c$ , hence  $G_A = (0, c(s-c), b(s-b))$  with similar expressions for  $G_B, G_C$ . It now follows that the coordinates of  $G$  are  $\left( \frac{1}{a(s-a)}, \frac{1}{b(s-b)}, \frac{1}{c(s-c)} \right)$  and by proposition 1, its trilinear polar is  $[a(s-a), b(s-b), c(s-c)] = m$  as claimed.

Roland H. Eddy, Memorial University of Newfoundland,  
St. John's, Newfoundland, Canada

## REFERENCES

- 1 Baptist P.: Über Nagelsche Punktpaare, *Mathematische Semesterberichte* 1, 118–126 (1988).
- 2 Coxeter H. S. M.: *The Real Projective Plane*, 2nd ed., Cambridge University Press, 1961.
- 3 Johnson R. A.: *Advanced Euclidean Geometry (Modern Geometry)*, Dover, NY (1969).
- 4 Nagel C. H. v.: *Untersuchungen über die wichtigsten zum Dreiecke gehörenden Kreise*, Leipzig (1836).
- 5 Sommerville D. M. Y.: *Analytical Conics*, G. Bell and Sons Ltd., London (1956).
- 6 Springer C. E.: *Geometry and Analysis of Projective Spaces*, W. H. Freeman, San Francisco (1964).

## NOTE

[1\*] The "prime" notation is particularly useful here since  $C_L$  and  $C'_L$  are related by the harmonic conjugacy involution, see [2].