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also modulo  $\frac{1}{2}\pi$  reduziert = 0, beträgt. Es bleiben die Terme längs  $A_0 A_3$  (mit dem Diederwinkel  $\frac{1}{2}\pi - \alpha$  und der Kantenlänge  $l = \int_{2 \sin \alpha}^u 1/z \, dz$  zwischen den gelegten Horosphären), und längs  $A_0 A_1$  (mit dem Diederwinkel  $\alpha$  und der Länge  $\int_1^u 1/z \, dz$  ausserhalb der Horosphären). Die elementare Ermittlung der Integrale führt auf

$$\Psi(L(\alpha)) = (\log u - \log |2 \sin \alpha|) \otimes (\frac{\pi}{2} - \alpha) + (\log u) \otimes \alpha,$$

also nach Reduzierung mod  $\frac{1}{2}\pi$  und Addition auf

$$\Psi(L(\alpha)) = \log |2 \sin \alpha| \otimes \alpha, \quad \text{w.z.b.w.}$$

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#### LITERATURVERZEICHNIS

- 1 Böhm J. und Schwulow H.: Eine Zerlegung von vierdimensionalen euklidischen und nichteuklidischen Simplexen in Orthoscheme. *Wiss. Z. Friedrich-Schiller-Univ. Jena Math. Natur. Reihe* 31, 545–555 (1982); MR 84 d: 52008.
- 2 Böhm J. und Hertel E.: *Polyedergeometrie in n-dimensionalen Räumen konstanter Krümmung*. Birkhäuser, Basel-Boston-Stuttgart 1981; MR 82 k: 52001 a.
- 3 Debrunner H. E.: *Dissecting orthoschemes into orthoschemes*, im Druck.
- 4 Dupont J. L. and Sah C.-H.: Scissors congruences, II. *J. Pure Appl. Algebra* 25, 159–195 (1982); 30 217 (1983); MR 84 b: 53062 b.
- 5 Hilbert D. und Cohn-Vossen S.: *Anschauliche Geometrie*. Springer, Berlin 1932.
- 6 Lobatschewskij N. I.: *Über die Anfangsgründe der Geometrie*, Kasaner Bote 1830. Übersetzt und kommentiert von F. Engel, Teubner, Leipzig 1898.
- 7 Milnor J. W.: *Hyperbolic Geometry: the first 150 years*. *Bull. Amer. Math. Soc.* 6, 9–24 (1982); MR 82 m: 57005.
- 8 Sah C.-H.: Scissors congruences, I. *The Gauss-Bonnet map*, *Math. Scand* 49, 181–210 (1982); 53, 62 (1983); MR 84 b: 53062 a.
- 9 Schläfli L.: *Theorie der vielfachen Kontinuität*, 1852. In: L. Schläfli, *Gesammelte Mathematische Abhandlungen*, Band I, 177–392, Birkhäuser, Basel 1950.
- 10 Wythoff W. A.: *The rule of Neper in the four dimensional space*. *K. Akad. Wet. Amsterdam, Proc. Sect. of Sci.* 9, 529–534 (1907).

## On cubic polynomials giving many primes

### 1. Introduction

In the following «prime» means positive or negative prime, that is an integer of the sequence ... -7, -5, -3, -2, 2, 3, 5, 7, 11 ...

The question of finding quadratic polynomials giving many primes has been widely investigated (see [2, p. 115–117] and [4, p. 141–143]) and we can cite for example  $2n^2 - 199$  (Karst, 1973, see [3]) giving 87 primes for  $n = 0, 1, \dots, 99$  and  $n^2 + n + 41$  (Euler, 1772) giving 86 primes in the same range.

The purpose of this paper is to exhibit some polynomials  $an^3 + bn^2 + cn + d$  giving many primes and to explain the way they were discovered. In the mathematical literature we found only one result about this subject in [1, p. 420]: «Escott found that  $x^3 + x^2 + 17$  is a prime for  $-14, -13, \dots, +10$ » (In fact this is not true since making  $x = -3$  yields the number  $-1$ ).

Arbitrarily we shall say that

« $p(n) = an^3 + bn^2 + cn + d$  gives many primes»

if and only if:

Card  $\{n \in [0, 1, 2, \dots, 99]$  such that  $p(n)$  is prime $\} \geq 75$ .

## 2. Method of computation

All calculations were performed with a micro-computer, using PASCAL and assembly language.

### *First step*

We determine all polynomials  $p(n) = an^3 + bn^2 + cn + d$  satisfying:

- (i)  $a = 1$  or  $a = 2$ .
- (ii)  $-1 \leq b \leq 1$  if  $a = 1$ ;  
 $-2 \leq b \leq 3$  if  $a = 2$ .
- (iii)  $|c| \leq 15$ .
- (iv)  $p(n) \not\equiv 0 \pmod{m}$  ( $m = 2, 3, 5, 7$ ;  $n \in \mathbb{N}$ ).
- (v)  $|d| \leq 10\,000$ .
- (vi) Card  $\{n \in [0, 1, \dots, 99]$  such that  $p(n)$  is prime $\} \geq 60$ .

### *Comments*

1) Condition (ii) is not a real restriction since replacing  $n$  by  $n + 1$  transforms  $p(n)$  into  $an^3 + (b + 3a)n^2 + (c + 2b + 3a)n + (d + a + b + c)$  showing that  $b$  can be reduced mod  $3a$ .

2) Without condition (iv) we should have 5,219,739 polynomials to examine, that is 521,973,900 primality tests to perform. With this condition the amount of computation is reduced by a factor 40 and the percentage of interesting polynomials lost is very low.

Practically, for given  $a$ ,  $b$ , and  $c$ , to get  $d \in [0, \dots, 209]$  ( $210 = 2 \cdot 3 \cdot 5 \cdot 7$ ) such that  $an^3 + bn^2 + cn + d$  is never divisible by 2, 3, 5 or 7 we run the following algorithm:

```

For  $p = 2, 3, 5$  and  $7$  do
  for  $n = 0$  to  $p - 1$  do
    for  $d = 0$  to  $p - 1$  do
      if  $(an^3 + bn^2 + cn + d) \bmod p = 0$ 
        then for  $j = d$  to  $209$  step  $p$  do suppress  $j$ .

```

The convenient values of  $d \pmod{210}$  are the remaining  $j \in [0, \dots, 209]$ .

*Example:* For  $a = 2$ ,  $b = 1$  and  $c = -7$ , we have only to consider  $d = 41, 107, 167, 191 \pmod{210}$ .

### *Second step*

For every polynomial provided by step 1:

- We search in the interval  $[-100, \dots, 100]$  a sequence of 100 consecutive integers  $j, j + 1, \dots, j + 99$  satisfying.  
Card  $\{n \in [j, j + 1, \dots, j + 99] \text{ such that } p(n) \text{ is prime}\}$  is maximum.
- If the maximum is greater than or equal to 75, changing  $n$  into  $n + j$  in  $p(n)$  yields a new polynomial giving at least 75 primes values for  $n = 0, 1, \dots, 99$ .
- If the maximum is less than 75, the polynomial is rejected.

### *Third step*

In the list of polynomials we get from step 2, there are pairs of polynomials  $(p(n), q(n))$  such that  $p(n) \equiv -q(-n - k)$  for some integer  $k$ . Geometrically this means that graphs of  $p$  and  $q$  are symmetric with respect to a vertical axis. For every pair of such polynomials we cancel one of them.

## 3. Results

From step 1 we get 215 polynomials. After step 2 it remains 34 polynomials and after step 3, 21 polynomials giving at least 75 prime values for  $n = 0, 1, \dots, 99$ . They are listed below with the number of prime values they have for  $0 \leq n < 100, 200, \dots, 500$ .

Polynomials from step 1 have also been exploited to search the longest sequences of consecutive integers  $n$  such that all corresponding values of  $p(n)$  are primes:

- Formulas (14) and (15) give 26 primes respectively for  $n = 16, \dots, 41$  and  $n = 74, \dots, 99$ ;
- Formula  $|2n^3 - 83n^2 + 1157n - 4999|$  gives 34 distinct primes for  $n = 0, \dots, 33$ .

No	a	b	c	d	Number of primes given by $an^3 + bn^2 + cn + d$ for				
					$n < 100$	$n < 200$	$n < 300$	$n < 400$	$n < 500$
(1)	1	-220	16 119	-392 723	75	134	179	219	261
(2)	1	-199	13 190	-290 869	75	124	163	206	235
(3)	1	-160	8 547	-142 811	75	130	179	221	264
(4)	1	-159	8 420	-148 153	76	124	164	203	238
(5)	1	-151	7 610	-129 097	76	125	168	204	245
(6)	1	-150	7 493	-124 277	76	128	170	217	250
(7)	1	-137	6 270	-95 203	75	121	168	197	233
(8)	1	-125	5 196	-73 291	79	130	174	212	254
(9)	1	-119	4 718	-71 741	75	118	158	190	224
(10)	1	-114	4 343	-54 829	76	125	171	200	232
(11)	1	-111	4 100	-49 367	76	119	150	199	246
(12)	1	-97	3 126	-32 603	75	115	161	195	235
(13)	1	-96	3 059	-32 563	75	127	162	192	225
(14)	1	-82	2 237	-20 407	75	112	149	183	218
(15)	2	-489	39 847	-1 084 553	75	134	176	222	267
(16)	2	-372	23 050	-476 027	75	128	174	211	239
(17)	2	-292	14 202	-231 551	76	124	160	206	252
(18)	2	-289	13 917	-221 891	75	124	172	210	247
(19)	2	-281	13 157	-204 487	78	129	169	214	250
(20)	2	-280	13 072	-203 857	76	123	168	204	240
(21)	2	-199	6 595	-79 657	79	125	174	217	257

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REFERENCES

- 1 Dickson L. E.: History of the theory of numbers, vol. I. Carnegie Inst. Washington, Chelsea, New York 1952.
- 2 Guy R. K.: Reviews in number theory 1973–1983, vol. I. AMS, Providence RI, 1984.
- 3 Karst E.: New quadratic forms with high density of primes. *Elem. Math.* 28, 116–118 (1973).
- 4 Leveque W. J.: Reviews in number theory 1940–1972, vol. I. AMS, Providence RI, 1974.

## A remark on the gamma function

According to Problem 188, Part II of G. Pólya and G. Szegő [2] for each integrable function  $f(x)$  on  $0 \leq x \leq 1$  we have:

$$\lim_{n \rightarrow \infty} \frac{1}{\varphi(n)} \sum_{\substack{k=1 \\ (k,n)=1}}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx,$$