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A pair of general triangle inequalities

Dedicated to Murray S. Klamkin

In this note the two "dual" triangle inequalities

$$(1/a + 1/b + 1/c)(R_1 + R_2 + R_3) > \begin{cases} 5 & \text{if } \max(A, B, C) \geq 120^\circ \\ 4 + 2/\sqrt{3} & \text{otherwise} \end{cases}$$

and

$$(a + b + c)(1/R_1 + 1/R_2 + 1/R_3) > 2(3 + 2\sqrt{2})$$

are proved. (R_1, R_2, R_3 represent the distances from P, a point of the interior or boundary of triangle ABC, to its vertices A, B and C, resp.) All numerical bounds cannot be improved.

1. Introduction

In this note in section 4 we prove two "dual" inequalities linking the sides and the distances from arbitrary interior or boundary point to the vertices of a triangle.
Before, in section 3, we give certain preliminary lemmata.
Finally, in section 5 we apply the results to a special point.

2. Notation

As usual for triangles, a, b, c denote the sides, A, B, C the vertices, r, R, s the incircle, circumcircle and semiperimeter, resp., and R_1, R_2, R_3 the distances from P, the point in question, to the vertices A, B and C, resp.

3. Lemmata

We start by proving the following inequality which is of interest for itself.

Lemma 1. Let $0 \leq x, y, z < 1$ be real numbers such that $x + y + z = 1$. Then

$$1/\sqrt{x^2 + xy + y^2} + 1/\sqrt{y^2 + yz + z^2} + 1/\sqrt{z^2 + zx + x^2} \geq 4 + 2/\sqrt{3}. \qquad (1)$$

Proof. We set $w_1 = \sqrt{x^2 + xy + y^2}$, $w_2 = \sqrt{y^2 + yz + z^2}$, $w_3 = \sqrt{z^2 + zx + x^2}$ and thus have to discuss the function $F(x, y, z, \lambda) = 1/w_1 + 1/w_2 + 1/w_3 + \lambda \cdot (x + y + z - 1)$ by the method of Lagrange-multipliers.

Necessary conditions for critical points with $0 < x, y, z < 1$ are

$$\partial F/\partial x = (-1/2) \cdot \{(2x + y)/w_1^3 + (2x + z)/w_3^3\} + \lambda = 0 \tag{2}$$

$$\partial F/\partial y = (-1/2) \cdot \{(2y + x)/w_1^3 + (2y + z)/w_2^3\} + \lambda = 0 \tag{3}$$

$$\partial F/\partial z = (-1/2) \cdot \{(2z + x)/w_3^3 + (2z + y)/w_2^3\} + \lambda = 0 \tag{4}$$

Adding the equations (2), (3) and (4) we get

$$\lambda = \{(x + y)/w_1^3 + (y + z)/w_2^3 + (z + x)/w_3^3\}/2.$$

Inserting this expression in (2) we obtain (as $y + z = 1 - x$) the relation

$$x \cdot \sum 1/w_i^3 = 1/w_2^3. \tag{5}$$

Similarly, from (3) and (4) there follow

$$y \cdot \sum 1/w_i^3 = 1/w_3^3 \quad \text{and} \tag{6}$$

$$z \cdot \sum 1/w_i^3 = 1/w_1^3 \quad \text{resp.} \tag{7}$$

Coupling (5), (6) and (6), (7) we get

$$x^2(y^2 + yz + z^2)^3 = y^2(z^2 + zx + x^2)^3 \tag{8}$$

$$y^2(z^2 + zx + x^2)^3 = z^2(x^2 + xy + y^2)^3. \tag{9}$$

As inequality (1) is symmetric we now may and do assume $0 < x \le y \le z < 1$. We put $y = ax$ and $z = bx$ where $1 \le a \le b$.
Then (9) becomes

$$a^2(b^2 + b + 1)^3 = b^2(a^2 + a + 1)^3. \tag{10}$$

By differentiation it is easily checked that the function $f(t) = (t^2 + t + 1)^3/t^2$ strictly increases for $t \ge 1$.
Thus (10) yields $a = b$, i.e. $y = z$, as necessary for critical points in the interior of the considered region.
Inserting this in (1) we have to prove

$$2/w_1 + 1/(y\sqrt{3}) \ge 4 + 2/\sqrt{3} \quad \text{where} \quad x + 2y = 1; \quad \text{i.e.}$$
$$2/\sqrt{3y^2 - 3y + 1} + 1/(y\sqrt{3}) \ge 4 + 2/\sqrt{3} \tag{11}$$

where $1/3 \le y \le 1/2$ (since $x \le y \le z$).
The transformation $y = 1/2 - w$ changes (11) to

$$g(w) := 2\sqrt{3}/\sqrt{12w^2 + 1} + 1/(1 - 2w) \ge 2\sqrt{3} + 1 \tag{12}$$

where $0 \le w \le 1/6$. We have now that $g'(w) \gtreqless 0$ iff

$$l(w) := (12w^2 + 1)^{3/2} \gtreqless 12\sqrt{3}\,w(1 - 2w)^2 =: r(w).$$

Clearly, $l(w)$ is strictly convex and $r(w)$ is strictly concave on $[0, 1/6]$ and, since $l(0) > r(0)$, $l(1/6) = r(1/6) = 8\sqrt{3}/9$, and $r'(1/6) = 0$, we deduce the existence of a $w_0 \in (0, 1/6)$ such that $g(w)$ increases on $(0, w_0)$ and decreases on $(w_0, 1/6)$. This means that the absolute minimum of $g(w)$ on $[0, 1/6]$ is $\min(g(0), g(1/6))$, which readily proves (12).
For (1) to be proved we still have to consider the boundary of the region. Let e.g. $z = 0$. Then $y = 1 - x$ and (1) becomes

$$h(x) := 1/\sqrt{x^2 - x + 1} + 1/(1 - x) + 1/x \ge 4 + 2/\sqrt{3}. \tag{13}$$

By symmetry, we may and do restrict ourselves to the case $0 < x \le 1/2$. Since $h(1/2) = 4 + 2/\sqrt{3}$, we have only to show that $h(x)$ is falling on $(0, 1/2)$, i.e. that $h'(x) < 0$ on $(0, 1/2)$, i.e. that

$$(1 - x)^2 x^2 < 2(x^2 - x + 1)^{3/2} \quad \text{on} \quad (0, 1/2). \tag{14}$$

Putting $m := x(1 - x)$ and noting $0 < m < 1/4$ we deduce the validity of (14) from the easily verified inequality

$$m^4 < 4(1 - m)^3, \quad m \in (0, 1/4).$$ □

Next, we show the following distance-inequality for the incenter I.

Lemma 2. $1/AI + 1/BI + 1/CI \ge 9\sqrt{3}/2s$ with equality iff the triangle is equilateral. (15)

Proof. In [1], p. 23, the inequality

$$AI + BI + CI \le 2s/\sqrt{3} \tag{16}$$

is established.
As the harmonic mean is never greater than the arithmetic one we get from (16)

$$1/AI + 1/BI + 1/CI \ge 9/(AI + BI + CI) \ge 9\sqrt{3}/2s.$$

In all inequalities there occurs equality iff the triangle is equilateral. □
Finally we prove a distance-inequality for the feet of the angle-bisectors w_a, w_b and w_c.

Lemma 3. $1/w_c + 1/c_1 + 1/c_2 > 6/s$ (17)

where c_1, c_2 denote the distances from the foot of w_c to A and B resp. Similar inequalities hold for w_a and w_b.

Proof. From elementary geometry the following relations are well-known:

$$c_1 = ac/(a+b), \quad c_2 = bc/(a+b) \quad \text{and} \quad w_c = 2ab\cos(C/2)/(a+b).$$

Consequently, $w_c < 2ab/(a+b) \le (a+b)/2$.
For (17) we are thus done if we verify the sharper inequality

$$2(a+b+c)/(a+b) + (a+b)(1/c_1 + 1/c_2) + c(1/c_1 + 1/c_2) > 12, \quad \text{i.e.}$$

$$2c/(a+b) + \{(a+b)/a + (a+b)/b\}(a+b)/c + c(1/c_1 + 1/c_2) > 10. \tag{18}$$

As clearly $c = c_1 + c_2$, we get $c(1/c_1 + 1/c_2) \ge 4$.
Furthermore, also $(a+b)/a + (a+b)/b \ge 4$.
Therefore, (18) can be strengthened to

$$2c/(a+b) + 4(a+b)/c > 6 \tag{19}$$

Putting $x = c/(a+b)$ and noting that $x < 1$ and $x + 2/x > 3$ for $x < 1$ we infer (19). □

4. Main Results

We are now in the position to prove the announced general inequalities. Let P be a point of the interior or the boundary of a triangle ABC.

Theorem 1

i) If one angle of the triangle is not less than 120°, then

$$(1/a + 1/b + 1/c)(R_1 + R_2 + R_3) > 5. \tag{20}$$

ii) If all angles are less than 120°, then

$$(1/a + 1/b + 1/c)(R_1 + R_2 + R_3) > 4 + 2/\sqrt{3}. \tag{21}$$

Both bounds cannot be improved.

Proof.

i) From [2], item 12.55, it is known: If, say, $A \ge 120°$ then $R_1 + R_2 + R_3 \ge b + c$. Thus, (20) follows from

$$(1/a + 1/b + 1/c)(b+c) = (b+c)/a + (b+c)(1/b + 1/c) > 5$$

which clearly holds true.

ii) It is known (e.g. [3], chapter 3) that $R_1 + R_2 + R_3$ is minimal if P coincides with Torricelli's (or Fermat's) point, i.e. the point subtending 120° with each side.

For typographical convenience let be now and further on $x = R_1$, $y = R_2$ and $z = R_3$.
By the law of cosine we get $a = \sqrt{x^2 + xy + y^2}$, etc.
Therefore, inequality (21) follows from lemma 1.
Taking triangles with $a = b$, $c \approx 2a$ and $P = C$ shows, that "5" in (20) cannot be improved.
Similarly the bound for (21): Take triangles with $a = b$, $c \approx a\sqrt{3}$ and $P = C$. □

Next we prove the following "dual" of the previous theorem.

Theorem 2

$$(a + b + c)(1/R_1 + 1/R_2 + 1/R_3) > 2(3 + 2\sqrt{2}); \quad \text{i.e.}$$
$$1/R_1 + 1/R_2 + 1/R_3 > (3 + 2\sqrt{2})/s. \tag{22}$$

The bound cannot be improved.

Proof. Let $w = \measuredangle APB$, $u = \measuredangle BPC$ and $v = \measuredangle CPA$.
Then $0 \le u, v, w \le 180°$ and $u + v + w = 360°$.
We then get $c = \sqrt{y^2 + z^2 - 2yz\cos u}$ etc. For (22) we thus have to minimize the function

$$F(x, y, z, u, v, w, \lambda) := (\sqrt{x^2 + y^2 - 2xy\cos u} + \sqrt{y^2 + z^2 - 2yz\cos v} + \sqrt{z^2 + x^2 - 2zx\cos w})(1/x + 1/y + 1/z) - \lambda \cdot (u + v + w - 360°).$$

From $\partial F/\partial u = 0$, $\partial F/\partial v = 0$ and $\partial F/\partial w = 0$ we get immediately

$$yz\sin u/a = xz\sin v/b = xy\sin w/c.$$

Particularly, $x\sin v/b = y\sin u/a$. This means geometrically that P lies on the angle-bisector of C. (Indeed, the law of sines (applied to the triangles PBC and PCA) yields $\sin u/a = \sin(\measuredangle PBC)/z$ and $\sin v/b = \sin(\measuredangle PAC)/z$. Therefore, $x\sin(\measuredangle PBC) = y\sin(\measuredangle PAC)$, i.e. P has equal distances from the sides a and b.)
Similarly it follows that P is on the angle-bisectors of A and B. Therefore the only (interior) critical point for F is the incenter I. But from lemma 2 we have

$$1/AI + 1/BI + 1/CI > 9\sqrt{3}/2s$$

and as $9\sqrt{3}/2 > 3 + 2\sqrt{2}$, we are done.
For the boundary we have two cases.

i) e.g. $w = 180°$. Then P is on the side c. As before it can be shown, that the minimizing P lies on the angle-bisector of C.
In lemma 3 we proved already $1/AP + 1/BP + 1/CP > 6/s$. As $6 > 3 + 2\sqrt{2}$ we are done.

ii) e.g. $w = 0$. Then $u = v = 180°$. Let be $y \leq x$.
In this case we have to deal with the degenerated "triangle" having the sides $c = x - y$, $a = y + z$, $b = x + z$.
(22) then becomes

$$(x + z)(1/x + 1/y + 1/z) \geq 3 + 2\sqrt{2}. \tag{23}$$

As $y \leq x$, (23) follows from

$$(x + z)(2/x + 1/z) \geq 3 + 2\sqrt{2}, \quad \text{i.e.} \quad 2z/x + x/z \geq 2\sqrt{2}, \quad \text{i.e.} \quad (z\sqrt{2} - x)^2 \geq 0.$$

Triangles with $c \approx 0$, $a = b \approx 3 + 2\sqrt{2}$ and P such that $R_1 = R_2 \approx 2 + \sqrt{2}$, $R_3 \approx 1 + \sqrt{2}$ show that the bound in (22) cannot be improved. □

Remarks. 1) Comparing inequality (21) with [2], item 12.55, i.e.

$$R_1 + R_2 + R_3 \geq \{(a^2 + b^2 + c^2 + 4F\sqrt{3})/2\}^{1/2} \tag{24}$$

it should be noted, that no general order can be given for the bounds of $R_1 + R_2 + R_3$ in (21) and (24).
2) We leave it to the reader to derive inequalities obtained by application of inversion, reciprocation and/or isogonal conjugation to theorems 1 and 2 (see [4], [5] and [6]).

5. A Special Point

Lemma 2 already states a special result.
Let $P = G$ be the centroid. Then $R_1 = 2m_a/3$ etc., where m_a etc. are the medians. Theorem 1 then reads

$$(1/a + 1/b + 1/c)(m_a + m_b + m_c) > 15/2. \tag{25}$$

Applying the process of median-duality (see [6] or [7]; i.e. if $I(a, b, c, m_a, m_b, m_c) \geq 0$ is a valid triangle-inequality then so is $I(m_a, m_b, m_c, 3a/4, 3b/4, 3c/4) \geq 0$) we get from (25)

$$1/m_a + 1/m_b + 1/m_c > 5/s. \tag{26}$$

This inequality was posed as a problem by the second author (see [8]). Triangles with $c \approx 0$, $a = b$ show that the bound "5" in (26) cannot be improved. □

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Aufgaben

Aufgabe 981. Man beweise oder widerlege folgende Aussage: Das Polynom

$$f(x) = x^5 + x - t$$

ist über $\mathbb{Z}$ irreduzibel, wenn $t = \pm p^n$, p Primzahl, $n \in \mathbb{N}$ und $p^n > 2$.

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Lösung. Die Aussage ist wahr.

Beweis. Sei $t = \pm q$ und q eine Primzahlpotenz. Falls ein quadratisches Polynom das Polynom $x^5 + x - t$ teilt, so ist $t = \pm 1$. Der Ansatz

$$x^5 + x - t = (x^2 + a_1 x + a_0)(x^3 + b_2 x^2 + b_1 x + b_0), \quad a_0, a_1, b_0, b_1, b_2 \in \mathbb{Z}$$

führt nämlich durch Koeffizientenvergleich und Elimination von b_0, b_1, b_2 sofort auf

$$3 a_0 a_1^2 - a_1^4 - a_0^2 = 1 \quad \text{und} \quad a_0 a_1 (a_1^2 - 2 a_0) = t.$$

Deshalb sind a_0 und a_1 teilerfremde Teiler der Primzahlpotenz q, woraus $a_0 = \pm 1$ oder $a_1 = \pm 1$ folgt. In beiden Fällen schliesst man $t = \pm 1$.
Falls aber ein lineares Polynom $x^5 + x - t$ teilt, so besitzt $x^5 + x - t$ eine ganzzahlige Nullstelle, also $t = \xi^5 + \xi$ für einen Teiler ξ von t; insbesondere ist t gerade. Daraus folgt $t = \pm 2$.

Bemerkung. Die Polynome $x^5 + x \pm y$ sind Beispiele für den folgenden Satz von V. G. Sprindžuk (Reducibility of polynomials and rational points on algebraic curves, Sém de Théorie des Nombres, Prog. Math. 12 (1981), 287–309). Sei $f \in \mathbb{Z}[x, y]$, absolut irreduzibel (d.h. irreduzibel in $C[x, y]$), $\deg_x f \geqq 2$, $f(0,0) = 0$ und $\frac{\partial f}{\partial x}(0,0) \neq 0$; dann ist für fast alle Primzahlpotenzen q das Polynom $f(x, q)$ in $\mathbb{Z}[x]$ irreduzibel.

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