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# A pair of general triangle inequalities

**Dedicated to Murray S. Klamkin**

In this note the two “dual” triangle inequalities

$$(1/a + 1/b + 1/c)(R_1 + R_2 + R_3) > \begin{cases} 5 & \text{if } \max(A, B, C) \geq 120^\circ \\ 4 + 2/\sqrt{3} & \text{otherwise} \end{cases}$$

and

$$(a + b + c)(1/R_1 + 1/R_2 + 1/R_3) > 2(3 + 2\sqrt{2})$$

are proved. ( $R_1, R_2, R_3$  represent the distances from  $P$ , a point of the interior or boundary of triangle  $ABC$ , to its vertices  $A, B$  and  $C$ , resp.) All numerical bounds cannot be improved.

## 1. Introduction

In this note in section 4 we prove two “dual” inequalities linking the sides and the distances from arbitrary interior or boundary point to the vertices of a triangle. Before, in section 3, we give certain preliminary lemmata. Finally, in section 5 we apply the results to a special point.

## 2. Notation

As usual for triangles,  $a, b, c$  denote the sides,  $A, B, C$  the vertices,  $r, R, s$  the incircle, circumcircle and semiperimeter, resp., and  $R_1, R_2, R_3$  the distances from  $P$ , the point in question, to the vertices  $A, B$  and  $C$ , resp.

## 3. Lemmata

We start by proving the following inequality which is of interest for itself.

**Lemma 1.** Let  $0 \leq x, y, z < 1$  be real numbers such that  $x + y + z = 1$ . Then

$$1/\sqrt{x^2 + xy + y^2} + 1/\sqrt{y^2 + yz + z^2} + 1/\sqrt{z^2 + zx + x^2} \geq 4 + 2/\sqrt{3}. \quad (1)$$

*Proof.* We set  $w_1 = \sqrt{x^2 + xy + y^2}$ ,  $w_2 = \sqrt{y^2 + yz + z^2}$ ,  $w_3 = \sqrt{z^2 + zx + x^2}$  and thus have to discuss the function  $F(x, y, z, \lambda) = 1/w_1 + 1/w_2 + 1/w_3 + \lambda \cdot (x + y + z - 1)$  by the method of Lagrange-multipliers.

Necessary conditions for critical points with  $0 < x, y, z < 1$  are

$$\partial F/\partial x = (-1/2) \cdot \{(2x + y)/w_1^3 + (2x + z)/w_3^3\} + \lambda = 0 \quad (2)$$

$$\partial F/\partial y = (-1/2) \cdot \{(2y + x)/w_1^3 + (2y + z)/w_2^3\} + \lambda = 0 \quad (3)$$

$$\partial F/\partial z = (-1/2) \cdot \{(2z + x)/w_3^3 + (2z + y)/w_2^3\} + \lambda = 0 \quad (4)$$

Adding the equations (2), (3) and (4) we get

$$\lambda = \{(x + y)/w_1^3 + (y + z)/w_2^3 + (z + x)/w_3^3\}/2.$$

Inserting this expression in (2) we obtain (as  $y + z = 1 - x$ ) the relation

$$x \cdot \sum 1/w_i^3 = 1/w_2^3. \quad (5)$$

Similarly, from (3) and (4) there follow

$$y \cdot \sum 1/w_i^3 = 1/w_3^3 \quad \text{and} \quad (6)$$

$$z \cdot \sum 1/w_i^3 = 1/w_1^3 \quad \text{resp.} \quad (7)$$

Coupling (5), (6) and (6), (7) we get

$$x^2(y^2 + yz + z^2)^3 = y^2(z^2 + zx + x^2)^3 \quad (8)$$

$$y^2(z^2 + zx + x^2)^3 = z^2(x^2 + xy + y^2)^3. \quad (9)$$

As inequality (1) is symmetric we now may and do assume  $0 < x \leq y \leq z < 1$ . We put  $y = ax$  and  $z = bx$  where  $1 \leq a \leq b$ .

Then (9) becomes

$$a^2(b^2 + b + 1)^3 = b^2(a^2 + a + 1)^3. \quad (10)$$

By differentiation it is easily checked that the function  $f(t) = (t^2 + t + 1)^3/t^2$  strictly increases for  $t \geq 1$ .

Thus (10) yields  $a = b$ , i.e.  $y = z$ , as necessary for critical points in the interior of the considered region.

Inserting this in (1) we have to prove

$$\begin{aligned} 2/w_1 + 1/(y\sqrt{3}) &\geq 4 + 2/\sqrt{3} \quad \text{where } x + 2y = 1; \quad \text{i.e.} \\ 2/\sqrt{3y^2 - 3y + 1} + 1/(y\sqrt{3}) &\geq 4 + 2/\sqrt{3} \end{aligned} \quad (11)$$

where  $1/3 \leq y \leq 1/2$  (since  $x \leq y \leq z$ ).

The transformation  $y = 1/2 - w$  changes (11) to

$$g(w) := 2\sqrt{3}/\sqrt{12w^2 + 1} + 1/(1 - 2w) \geq 2\sqrt{3} + 1 \quad (12)$$

where  $0 \leq w \leq 1/6$ . We have now that  $g'(w) \geq 0$  iff

$$l(w) := (12w^2 + 1)^{3/2} \geq 12\sqrt{3}w(1 - 2w)^2 =: r(w).$$

Clearly,  $l(w)$  is strictly convex and  $r(w)$  is strictly concave on  $[0, 1/6]$  and, since  $l(0) > r(0)$ ,  $l(1/6) = r(1/6) = 8\sqrt{3}/9$ , and  $r'(1/6) = 0$ , we deduce the existence of a  $w_0 \in (0, 1/6)$  such that  $g(w)$  increases on  $(0, w_0)$  and decreases on  $(w_0, 1/6)$ . This means that the absolute minimum of  $g(w)$  on  $[0, 1/6]$  is  $\min(g(0), g(1/6))$ , which readily proves (12).

For (1) to be proved we still have to consider the boundary of the region. Let e.g.  $z = 0$ . Then  $y = 1 - x$  and (1) becomes

$$h(x) := 1/\sqrt{x^2 - x + 1} + 1/(1 - x) + 1/x \geq 4 + 2/\sqrt{3}. \quad (13)$$

By symmetry, we may and do restrict ourselves to the case  $0 < x \leq 1/2$ . Since  $h(1/2) = 4 + 2/\sqrt{3}$ , we have only to show that  $h(x)$  is falling on  $(0, 1/2)$ , i.e. that  $h'(x) < 0$  on  $(0, 1/2)$ , i.e. that

$$(1 - x)^2 x^2 < 2(x^2 - x + 1)^{3/2} \quad \text{on } (0, 1/2). \quad (14)$$

Putting  $m := x(1 - x)$  and noting  $0 < m < 1/4$  we deduce the validity of (14) from the easily verified inequality

$$m^4 < 4(1 - m)^3, \quad m \in (0, 1/4).$$

□

Next, we show the following distance-inequality for the incenter  $I$ .

**Lemma 2.**  $1/AI + 1/BI + 1/CI \geq 9\sqrt{3}/2s$  with equality iff the triangle is equilateral. (15)

*Proof.* In [1], p. 23, the inequality

$$AI + BI + CI \leq 2s/\sqrt{3} \quad (16)$$

is established.

As the harmonic mean is never greater than the arithmetic one we get from (16)

$$1/AI + 1/BI + 1/CI \geq 9/(AI + BI + CI) \geq 9\sqrt{3}/2s.$$

In all inequalities there occurs equality iff the triangle is equilateral. □

Finally we prove a distance-inequality for the feet of the angle-bisectors  $w_a$ ,  $w_b$  and  $w_c$ .

**Lemma 3.**  $1/w_c + 1/c_1 + 1/c_2 > 6/s$  (17)

where  $c_1, c_2$  denote the distances from the foot of  $w_c$  to  $A$  and  $B$  resp. Similar inequalities hold for  $w_a$  and  $w_b$ .

*Proof.* From elementary geometry the following relations are well-known:

$$c_1 = a c / (a + b), \quad c_2 = b c / (a + b) \quad \text{and} \quad w_c = 2 a b \cos(C/2) / (a + b).$$

Consequently,  $w_c < 2 a b / (a + b) \leq (a + b)/2$ .

For (17) we are thus done if we verify the sharper inequality

$$\begin{aligned} 2(a + b + c)/(a + b) + (a + b)(1/c_1 + 1/c_2) + c(1/c_1 + 1/c_2) &> 12, \quad \text{i.e.} \\ 2c/(a + b) + \{(a + b)/a + (a + b)/b\}(a + b)/c + c(1/c_1 + 1/c_2) &> 10. \end{aligned} \quad (18)$$

As clearly  $c = c_1 + c_2$ , we get  $c(1/c_1 + 1/c_2) \geq 4$ .

Furthermore, also  $(a + b)/a + (a + b)/b \geq 4$ .

Therefore, (18) can be strengthened to

$$2c/(a + b) + 4(a + b)/c > 6 \quad (19)$$

Putting  $x = c/(a + b)$  and noting that  $x < 1$  and  $x + 2/x > 3$  for  $x < 1$  we infer (19).  $\square$

#### 4. Main Results

We are now in the position to prove the announced general inequalities. Let  $P$  be a point of the interior or the boundary of a triangle  $ABC$ .

##### Theorem 1

i) If one angle of the triangle is not less than  $120^\circ$ , then

$$(1/a + 1/b + 1/c)(R_1 + R_2 + R_3) > 5. \quad (20)$$

ii) If all angles are less than  $120^\circ$ , then

$$(1/a + 1/b + 1/c)(R_1 + R_2 + R_3) > 4 + 2/\sqrt{3}. \quad (21)$$

Both bounds cannot be improved.

*Proof.*

i) From [2], item 12.55, it is known: If, say,  $A \geq 120^\circ$  then  $R_1 + R_2 + R_3 \geq b + c$ . Thus, (20) follows from

$$(1/a + 1/b + 1/c)(b + c) = (b + c)/a + (b + c)(1/b + 1/c) > 5$$

which clearly holds true.

ii) It is known (e.g. [3], chapter 3) that  $R_1 + R_2 + R_3$  is minimal if  $P$  coincides with Torricelli's (or Fermat's) point, i.e. the point subtending  $120^\circ$  with each side.

For typographical convenience let be now and further on  $x = R_1$ ,  $y = R_2$  and  $z = R_3$ .

By the law of cosine we get  $a = \sqrt{x^2 + xy + y^2}$ , etc.

Therefore, inequality (21) follows from lemma 1.

Taking triangles with  $a = b$ ,  $c \approx 2a$  and  $P = C$  shows, that "5" in (20) cannot be improved.

Similarly the bound for (21): Take triangles with  $a = b$ ,  $c \approx a\sqrt{3}$  and  $P = C$ .  $\square$

Next we prove the following "dual" of the previous theorem.

## Theorem 2

$$(a + b + c)(1/R_1 + 1/R_2 + 1/R_3) > 2(3 + 2\sqrt{2}); \quad \text{i.e.}$$

$$1/R_1 + 1/R_2 + 1/R_3 > (3 + 2\sqrt{2})/s. \quad (22)$$

The bound cannot be improved.

*Proof.* Let  $w = \angle APB$ ,  $u = \angle BPC$  and  $v = \angle CPA$ .

Then  $0 \leq u, v, w \leq 180^\circ$  and  $u + v + w = 360^\circ$ .

We then get  $c = \sqrt{y^2 + z^2 - 2yz \cos u}$  etc. For (22) we thus have to minimize the function

$$F(x, y, z, u, v, w, \lambda) := (\sqrt{x^2 + y^2 - 2xy \cos u} + \sqrt{y^2 + z^2 - 2yz \cos v} + \sqrt{z^2 + x^2 - 2zx \cos w})(1/x + 1/y + 1/z) - \lambda \cdot (u + v + w - 360^\circ).$$

From  $\partial F / \partial u = 0$ ,  $\partial F / \partial v = 0$  and  $\partial F / \partial w = 0$  we get immediately

$$yz \sin u/a = xz \sin v/b = xy \sin w/c.$$

Particularly,  $x \sin v/b = y \sin u/a$ . This means geometrically that  $P$  lies on the angle-bisector of  $C$ . (Indeed, the law of sines (applied to the triangles  $PBC$  and  $PCA$ ) yields  $\sin u/a = \sin(\angle PBC)/z$  and  $\sin v/b = \sin(\angle PAC)/z$ . Therefore,  $x \sin(\angle PBC) = y \sin(\angle PAC)$ , i.e.  $P$  has equal distances from the sides  $a$  and  $b$ .)

Similarly it follows that  $P$  is on the angle-bisectors of  $A$  and  $B$ . Therefore the only (interior) critical point for  $F$  is the incenter  $I$ . But from lemma 2 we have

$$1/AI + 1/BI + 1/CI > 9\sqrt{3}/2s$$

and as  $9\sqrt{3}/2 > 3 + 2\sqrt{2}$ , we are done.

For the boundary we have two cases.

- i) e.g.  $w = 180^\circ$ . Then  $P$  is on the side  $c$ . As before it can be shown, that the minimizing  $P$  lies on the angle-bisector of  $C$ .

In lemma 3 we proved already  $1/AP + 1/BP + 1/CP > 6/s$ . As  $6 > 3 + 2\sqrt{2}$  we are done.

ii) e.g.  $w = 0$ . Then  $u = v = 180^\circ$ . Let be  $y \leq x$ .

In this case we have to deal with the degenerated “triangle” having the sides  $c = x - y$ ,  $a = y + z$ ,  $b = x + z$ .

(22) then becomes

$$(x + z)(1/x + 1/y + 1/z) \geq 3 + 2\sqrt{2}. \quad (23)$$

As  $y \leq x$ , (23) follows from

$$(x + z)(2/x + 1/z) \geq 3 + 2\sqrt{2}, \text{ i.e. } 2z/x + x/z \geq 2\sqrt{2}, \text{ i.e. } (z\sqrt{2} - x)^2 \geq 0.$$

Triangles with  $c \approx 0$ ,  $a = b \approx 3 + 2\sqrt{2}$  and  $P$  such that  $R_1 = R_2 \approx 2 + \sqrt{2}$ ,  $R_3 \approx 1 + \sqrt{2}$  show that the bound in (22) cannot be improved.  $\square$

*Remarks.* 1) Comparing inequality (21) with [2], item 12.55, i.e.

$$R_1 + R_2 + R_3 \geq \{(a^2 + b^2 + c^2 + 4F\sqrt{3})/2\}^{1/2} \quad (24)$$

it should be noted, that no general order can be given for the bounds of  $R_1 + R_2 + R_3$  in (21) and (24).

2) We leave it to the reader to derive inequalities obtained by application of inversion, reciprocation and/or isogonal conjugation to theorems 1 and 2 (see [4], [5] and [6]).

## 5. A Special Point

Lemma 2 already states a special result.

Let  $P = G$  be the centroid. Then  $R_1 = 2m_a/3$  etc., where  $m_a$  etc. are the medians. Theorem 1 then reads

$$(1/a + 1/b + 1/c)(m_a + m_b + m_c) > 15/2. \quad (25)$$

Applying the process of median-duality (see [6] or [7]; i.e. if  $I(a, b, c, m_a, m_b, m_c) \geq 0$  is a valid triangle-inequality then so is  $I(m_a, m_b, m_c, 3a/4, 3b/4, 3c/4) \geq 0$ ) we get from (25)

$$1/m_a + 1/m_b + 1/m_c > 5/s. \quad (26)$$

This inequality was posed as a problem by the second author (see [8]). Triangles with  $c \approx 0$ ,  $a = b$  show that the bound “5” in (26) cannot be improved.  $\square$

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## Aufgaben

**Aufgabe 981.** Man beweise oder widerlege folgende Aussage: Das Polynom

$$f(x) = x^5 + x - t$$

ist über  $\mathbb{Z}$  irreduzibel, wenn  $t = \pm p^n$ ,  $p$  Primzahl,  $n \in \mathbb{N}$  und  $p^n > 2$ .

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**Lösung.** Die Aussage ist wahr.

*Beweis.* Sei  $t = \pm q$  und  $q$  eine Primzahlpotenz. Falls ein quadratisches Polynom das Polynom  $x^5 + x - t$  teilt, so ist  $t = \pm 1$ . Der Ansatz

$$x^5 + x - t = (x^2 + a_1 x + a_0)(x^3 + b_2 x^2 + b_1 x + b_0), \quad a_0, a_1, b_0, b_1, b_2 \in \mathbb{Z}$$

führt nämlich durch Koeffizientenvergleich und Elimination von  $b_0, b_1, b_2$  sofort auf

$$3a_0a_1^2 - a_1^4 - a_0^2 = 1 \quad \text{und} \quad a_0a_1(a_1^2 - 2a_0) = t.$$

Deshalb sind  $a_0$  und  $a_1$  teilerfremde Teiler der Primzahlpotenz  $q$ , woraus  $a_0 = \pm 1$  oder  $a_1 = \pm 1$  folgt. In beiden Fällen schliesst man  $t = \pm 1$ .

Falls aber ein lineares Polynom  $x^5 + x - t$  teilt, so besitzt  $x^5 + x - t$  eine ganzzahlige Nullstelle, also  $t = \xi^5 + \xi$  für einen Teiler  $\xi$  von  $t$ ; insbesondere ist  $t$  gerade. Daraus folgt  $t = \pm 2$ .

*Bemerkung.* Die Polynome  $x^5 + x \pm y$  sind Beispiele für den folgenden Satz von V. G. Sprindžuk (Reducibility of polynomials and rational points on algebraic curves, Séminaire de Théorie des Nombres, Prog. Math. 12 (1981), 287–309). Sei  $f \in \mathbb{Z}[x, y]$ , absolut irreduzibel (d. h. irreduzibel in  $\mathbb{C}[x, y]$ ),  $\deg_x f \geq 2$ ,  $f(0, 0) = 0$  und  $\frac{\partial f}{\partial x}(0, 0) \neq 0$ ; dann ist für fast alle Primzahlpotenzen  $q$  das Polynom  $f(x, q)$  in  $\mathbb{Z}[x]$  irreduzibel.

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