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REFERENCES

1 Berger J. O.: Statistical decision theory and Bayesian analysis, 2nd ed., Springer, Berlin – Heidelberg – New York 1985.
2 Hodges J. L. and Lehmann E. L.: Some problems in minimax point estimation. Ann. Math. Statist. *21*, 182–197 (1950).
3 Lehmann E. L.: Theory of point estimation. Wiley, New York 1983.
4 Loeffel H.: Statistische Inferenz und strategisches Spiel. El. Math. *40*, 109–120 (1985).
5 Neumann J. v.: Zur Theorie der Gesellschaftsspiele. Math. Annalen *100*, 295–320 (1928).
6 Rauhut B., Schmitz N. und Zachow E.-W.: Spieltheorie. Teubner, Stuttgart 1979.

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Some integral inequalities

The aim of this note is to prove some integral inequalities and to find interesting applications for the logarithmic and exponential functions. These relations have some known corollaries ([3], [4], [5], [8]).

Theorem 1. *Let* $f:[a,b]\to\mathbb{R}$ $(a<b)$ *be a differentiable function with increasing (strictly increasing) derivative on* $[a,b]$. *Then one has the following inequalities:*

$$\int_a^b f(t)\,dt \underset{(>)}{\geq} (b-a)\, f\left(\frac{a+b}{2}\right) \tag{1}$$

$$2\cdot\int_a^b f(t)\,dt \underset{(<)}{\leq} (b-a)\, f(\sqrt{ab}) + (\sqrt{b}-\sqrt{a})(\sqrt{b}\, f(b) + \sqrt{a}\, f(a))$$

(Here $0\leq a<b$*).* (2)

Proof. The Lagrange mean-value theorem implies: $f(y)-f(x) \underset{(>)}{\geq} (y-x)\,f'(x)$ for all $x,y\in[a,b]$. Take $x=(a+b)/2$ and integrate the obtained inequality:

$$\int_a^b f(y)\,dy-(b-a)\,f\left(\frac{a+b}{2}\right) \underset{(>)}{\geq} f'\left(\frac{a+b}{2}\right)\cdot\int_a^b\left(y-\frac{a+b}{2}\right)dy=0,$$

i.e. relation (1).

In order to prove (2) consider as above the inequality $f(y)-f(x) \underset{(<)}{\leq} (y-x)\,f'(y)$ with $x=\sqrt{ab}$. Integrating by parts on $[a,b]$ we get

$$\int_a^b f(y)\,dy-(b-a)\,f(\sqrt{ab}) \underset{(<)}{\leq} (y-\sqrt{ab})\,f(y)\Big|_a^b-\int_a^b f(y)\,dy$$

which easily implies (2).

Remark. Inequality (1) is called sometimes "Hadamard's inequality" and it is valid for convex functions f as well with the same proof, but using $f'_+\left(\frac{a+b}{2}\right)$ instead of $f'\left(\frac{a+b}{2}\right)$ (see also [1]).

In applications is useful the following generalization of (1) (see [9])

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a 2 k-times differentiable function, having continuous 2 k-th derivative on $[a,b]$ and satisfying $f^{(2k)}(t)\underset{(>)}{\geq}0$ for $t\in(a,b)$. Then one has the inequality:*

$$\int_a^b f(t)\,dt \underset{(>)}{\geq} \sum_{p=1}^{k} \frac{(b-a)^{2p-1}}{2^{2p-2}(2p-1)!} f^{(2p-2)}\left(\frac{a+b}{2}\right). \tag{3}$$

Proof. Apply Taylor's formula (with Lagrange remainder term) for f around $\left(\frac{a+b}{2}\right)$ and integrate term by term this relation. Remarking that $\int_a^b \left(x-\frac{a+b}{2}\right)^{2m-1} dx = 0$ for $m=1,2,3,\ldots$, we obtain

$$\int_a^b f(x)\,dx = (b-a)\,f\left(\frac{a+b}{2}\right) + \frac{(b-a)}{2^2\,3!} f''\left(\frac{a+b}{2}\right)$$

$$+\ldots+\frac{(b-a)^{2k-1}}{2^{2k-2}(2k-1)!} f^{(2k-2)}\left(\frac{a+b}{2}\right) + \int_a^b \frac{(x-(a+b)/2)^{2k}}{(2k)!} f^{(2k)}(\xi)\,dx.$$

Taking into account $f^{(2k)}(\xi)\underset{(>)}{\geq}0$, we get the desired inequality (3).

Applications. 1) Let $a>0$, $b=a+1$, $f_1(t)=\frac{1}{t}$ and $f_2(t)=-\ln t$ in (1). We can easily deduce the following double inequality:

$$\frac{2a+2}{2a+1} < \frac{e}{\left(1+\frac{1}{a}\right)^a} < \sqrt{1+\frac{1}{a}} \tag{4}$$

containing inequalities studied by E. R. Love [4] and G. Pólya – G. Szegö [7]. Using Bernoulli's inequality we have $(1+1/(2a+1))^{5/2} > 1+5/(4a+2) \geq 1+1/a$, for $a\geq 2$. Hence we have:

$$\left(1+\frac{1}{a}\right)^{a+2/5} < e < \left(1+\frac{1}{a}\right)^{a+1/2} \quad (a\geq 2) \tag{5}$$

2) By repeating the same argument in (3) for $k = 2$, $b = a + 1\,(a > 0)$, $f_1(t) = \frac{1}{t}$, $f_2(t) = -\ln t$, we obtain:

$$\frac{2a+2}{2a+1} e^{\frac{1}{6(2a+1)^2}} < \frac{e}{\left(1+\frac{1}{a}\right)^a} < \sqrt{1+\frac{1}{a}} \cdot e^{\frac{1}{3(2a+1)^2}}. \tag{6}$$

This inequality implies for $a > 0$ e.g. that

$$e^{\frac{1}{2a}\left(1-\frac{1}{a}\right)} < \frac{e}{\left(1+\frac{1}{a}\right)^a} < e^{\frac{1}{2a}\left(1-\frac{1}{2a}\right)} \tag{7}$$

and so

$$A_n = \left(\frac{1}{2} - n \ln e \Big/ \left(1+\frac{1}{n}\right)^n\right) = 0(1/n)$$

which can be compared with the more familiar $\lim_{n\to\infty} A_n = 0$.

3) Apply (1), (2) for $f(t) = \frac{1}{t}$ to deduce

$$\sqrt{ab} < L(a,b) < \frac{a+b}{2}, \tag{8}$$

Where $L(a,b) = \frac{b-a}{\ln b - \ln a}$ denotes the logarithmic means (see [2], [3]). The right-hand side of this inequality is due to B. Ostle and H. L. Terwilliger [6]. The left-hand inequality was stated by B. C. Carlson [2]. (8) was rediscovered also by A. Lupaş [5].

4) Select $f(t) = -\ln t$ in (2). This application yields the following improvement of the right-hand side of (8):

$$L(a,b) < \left(\frac{a+b}{2} + \sqrt{ab}\right)/2. \tag{9}$$

5) An interesting remark is that one can use (8) (and also (9)) to obtain refinements of this inequality. Indeed, let us consider $a = \sqrt{x}$, $b = \sqrt{y}$ in (8). It follows that

$$\sqrt{xy} < \sqrt[4]{xy}\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right) < L(x,y) < \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 < \frac{x+y}{2}. \tag{10}$$

With the same argument we can derive (on base of (9)):

$$L(x,y) < \frac{1}{2}\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 + \frac{1}{4}(\sqrt{x}+\sqrt{y})\sqrt[4]{xy}. \tag{11}$$

6) In order to arrive to a better refinement, we can consider the relation (3) for $f(t) = 1/t$, $k = 2 (0 < a < b)$. It results $L(a,b) < \frac{3}{8}(a+b)^3/(a^2+ab+b^2)$. Letting $a = \sqrt[3]{x}, b = \sqrt[3]{y}$, this is just one of the Lin [3] and Rüthing [8] inequalities:

$$L(x,y) < \left(\frac{\sqrt[3]{x}+\sqrt[3]{y}}{2}\right)^3 \tag{12}$$

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REFERENCES

1 Beckenbach E. F.: Convex functions. Bull. Amer. Math. Soc. *54*, 439–460 (1948).
2 Carlson B. C.: Some inequalities for hypergeometric functions. Proc. Amer. Math. Soc. *17*, 32–39 (1966).
3 Lin T. P.: The power mean and the logarithmic mean. Amer. Math. Monthly *81*, 879–883 (1974).
4 Love E. R.: Some logarithm inequalities. Math. Gaz. *64*, 55–57 (1980).
5 Lupaş A.: Problem 12739. Gaz. Mat. (Bucureşti), Number 2, 1973.
6 Ostle B. and Terwilliger H. L.: A comparison of two means. Proc. Montana Acad. Sci. *17*, 69–70 (1957).
7 Pólya G. and Szegö G.: Aufgaben und Lehrsätze aus der Analysis, vol. I, Ch. 4, Springer 1924.
8 Rüthing D.: Eine allgemeine logarithmische Ungleichung. El. Math. *41*, 14–16 (1986).
9 Sándor J.: On Hadamard's inequality (Hungarian). Mat. Lapok (Cluj) *87*, 427–430 (1982).

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Kleine Mitteilungen

Eine komplexe Ungleichung aus elementarer Sicht

Aus der Operatorentheorie ist folgende Ungleichung bekannt:

Für $f(z_1, z_2) = 1 + 2(z_1 + z_2) + (z_1 - z_2)^2$ mit $z_1, z_2 \in C$ und $|z_1| = |z_2| = 1$ gilt

$$|f(z_1, z_2)| \le 5. \tag{1}$$

Man kann (1) mit Hilfe der Ableitungen beweisen, indem man $z_1 = \exp(i\varphi_1)$, $z_2 = \exp(i\varphi_2)$ setzt und dann die Paare φ_1, φ_2 sucht, für die

$$\frac{\partial}{\partial\varphi_1}|f(z_1, z_2)|^2 = \frac{\partial}{\partial\varphi_2}|f(z_1, z_2)|^2 = 0$$