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# On the incongruence of consecutive fourth powers

## 1. Introduction

In [1], Arnold, Benkoski and McCabe solve the following problem: Given an integer  $n \geq 1$ , what is the smallest positive integer  $k$  such that  $1^2, 2^2, \dots, n^2$  are all incongruent modulo  $k$ ? Let  $D(n)$  denote this integer; they show that

$$D(n) = \begin{cases} 1, & \text{if } n = 1, \\ 2, & \text{if } n = 2, \\ 9, & \text{if } n = 4, \\ \min \{k \mid k \geq 2n \text{ and } k = p \text{ or } 2p \text{ with } p \text{ prime}\}, & \text{for all other } n. \end{cases}$$

Their proof, based on Bertrand's postulate, is neat and elementary. Their problem suggests the following generalization. Given integers  $n \geq 1$  and  $j \geq 1$ , determine

$$D(j, n) := \min \{k \geq 1 \mid a^j \not\equiv b^j \pmod{k} \text{ if } 1 \leq a < b \leq n\}.$$

If  $j = 2$ ,  $D(j, n) = D(n)$ . In this article, we determine  $D(2^h, n)$  for all  $n \geq 1$  and  $h \geq 2$ . The proofs are elementary, and use an extension of Bertrand's postulate to primes  $p \equiv 3 \pmod{4}$ .

In the last section we mention what is known about  $D(j, n)$  for other values of  $j$ .

## 2. The Theorem

We shall prove the following

**Theorem.** *For  $h \geq 2$ , we have*

$$D(2^h, n) = \begin{cases} 1, & \text{if } n = 1, \\ 2, & \text{if } n = 2, \\ 9, & \text{if } n = 4, \\ 18, & \text{if } n = 8, \\ \min \{k \mid k \geq 2n \text{ and } k = p \text{ or } 2p, \text{ with } p \equiv 3 \pmod{4}\}, & \text{for all other } n. \end{cases} \tag{2.1}$$

Here and in the sequel,  $p$  denotes a prime; we will always assume  $h \geq 2$ . Following the notation in [1], we denote the quantity on the right side of (2.1) by  $B(2^h, n)$ . The proof that  $D(2^h, n) = B(2^h, n)$  proceeds by establishing the following five lemmas.

**Lemma 1.**  $B(2^h, n) < 4n$  for  $n \geq 1$ .

**Lemma 2.** If  $p > 2n$  and  $p \equiv 3 \pmod{4}$ , then  $D(2^h, n) \leq p$ .

**Lemma 3.** *If  $2p \geq 2n$  and  $p \equiv 3 \pmod{4}$ , then  $D(2^h, n) \leq 2p$ .*

**Lemma 4.**  *$D(2^h, n) \geq 2n$  for  $n \geq 3$ .*

**Lemma 5.** *If  $n \geq 5$ ,  $n \neq 8$  and  $2n \leq m < B(2^h, n)$ , then  $D(2^h, n) \neq m$ .*

### 3. The proofs

We shall use the following observation several times: if  $h \geq 2$ ,  $p \equiv 3 \pmod{4}$  and

$$a^{2^h} \equiv b^{2^h} \pmod{m}, \quad (3.1)$$

with  $m = p, 2p, p^2$  or  $2p^2$ , then

$$a^{2^{h-1}} \equiv b^{2^{h-1}} \pmod{m}, \quad (3.2)$$

and by induction,

$$a^2 \equiv b^2 \pmod{m}. \quad (3.3)$$

Indeed, consider the case  $m = p$ :

$$a^{2^h} - b^{2^h} = (a^{2^{h-1}} - b^{2^{h-1}})(a^{2^{h-1}} + b^{2^{h-1}}),$$

the second factor on the right side is a sum of two squares and  $p \equiv 3 \pmod{4}$ . Hence if  $p$  divides  $(a^{2^{h-1}} + b^{2^{h-1}})$ , then  $p|a$  and  $p|b$  [5, Theorem 367].

Similarly, if (3.1) holds  $\pmod{2p}$  then so does (3.2) because then  $a \equiv b \pmod{2}$ .

And if (3.1) holds with  $m = p^2$  or  $m = 2p^2$ , then (3.2) does also, since  $p^2 | a^{2^{h-1}}$  if  $p|a$  and  $h \geq 2$ .

**Proof of Lemma 1.** For  $1 \leq n \leq 4$  and  $n = 8$ ,  $B(2^h, n) < 4n$  by inspection. For  $n \geq 5$ ,  $n \neq 8$ , this inequality follows immediately from the existence, for each integer  $x \geq 4$ , of a prime  $p \equiv 3 \pmod{4}$  such that  $x < p < 2x$ . An elementary proof of this extension of Bertrand's postulate (and of similar results for other arithmetical progressions) was given by Erdős [4], after Breusch [3] had proved it using complex variable techniques.

**Proof of Lemma 2.** Suppose  $D(2^h, n) > p$ . Then, for some integers  $a$  and  $b$  with  $1 \leq a < b \leq n$ , (3.1) would hold with  $m = p$ . But then, since  $p \equiv 3 \pmod{4}$ , we also would have (3.3) with  $m = p$ . And this is impossible, because  $1 \leq b - a < a + b < 2n < p$ .

**Proof of Lemma 3.** If  $D(2^h, n) > 2p$  then (3.1) holds, with  $m = 2p$ , for some  $a, b$  with  $1 \leq a < b \leq n$ . But then  $a^2 \equiv b^2 \pmod{2p}$ , which is impossible since  $1 \leq b - a < a + b < 2n \leq 2p$ .

**Proof of Lemma 4.** Clearly,  $D(2^h, n) \geq 3$  if  $n \geq 3$ . Any integer  $k$  such that  $3 \leq k < 2n$  can be written  $k = a + b$ , with  $1 \leq a < b \leq n$ . Then  $a^{2^h} \equiv b^{2^h} \pmod{k}$ , since  $(a + b) | (a^{2^h} - b^{2^h})$ ; hence  $D(2^h, n) \neq k$ .

**Proof of Lemma 5.** By definition of  $B(4, n)$ , the assumption  $2n \leq m < B(4, n)$  entails  $m \neq p$  and  $m \neq 2p$ , if  $p \equiv 3 \pmod{4}$ . We accordingly have 5 possibilities, if  $n \geq 2$ :

- (1)  $m = rs, r > s \geq 2, r \equiv s \pmod{2}$ ,
- (2)  $m = 2rs, r > s \geq 3, r \equiv s \equiv 1 \pmod{2}$ ,
- (3)  $m = p$  or  $2p, p \equiv 1 \pmod{4}$ ,
- (4)  $m = p^2$ ,
- (5)  $m = 2p^2$ .

Since (3.2) implies (3.1) for any modulus  $m$  and any  $h \geq 1$ , it suffices to show that in each case there are integers  $a, b$  such that  $1 \leq a < b \leq n$  and  $a^2 \equiv b^2 \pmod{m}$  or  $a^4 \equiv b^4 \pmod{m}$ .

In cases (1) and (2), take  $a = \frac{1}{2}(r - s), b = \frac{1}{2}(r + s)$ . Then  $1 \leq a < b$  and  $a^2 \equiv b^2 \pmod{m}$ . Also,  $\frac{1}{2}(r + s) \leq \frac{1}{4}rs + 1$  since  $r \geq 2$  and  $s \geq 2$  (write  $2(r - s) \leq s(r - 2)$ ). Hence  $b \leq \frac{1}{4}rs + 1 \leq \frac{1}{4}m + 1$ ; since  $m < B(2^h, n) < 4n$ , we have  $b \leq n$ .

In case (3),  $m$  is a sum of two squares [5, Theorem 366], say  $m = a^2 + b^2$ , with  $1 \leq a < b$ . Further,  $b^2 < m < B(2^h, n) < 4n$ , whence  $b < n$  if  $n \geq 4$ . And  $m = a^2 + b^2$  implies  $a^4 \equiv b^4 \pmod{m}$ .

In case (4), take  $a = p, b = 2p$ . Then  $1 < a < b$  and  $a^2 \equiv b^2 \pmod{m}$ . Also,  $b = 2p = 2\sqrt{m} < 4\sqrt{n}$ , whence  $b < n$  if  $n \geq 16$ .

In case (5),  $a = p$  and  $b = 3p$  satisfy  $1 < a < b$  and  $a^2 \equiv b^2 \pmod{m}$ . And  $b = 3\sqrt{m/2} < \sqrt{18n}$ , whence  $b < n$  if  $n \geq 18$ .

In order to complete the discussion of cases (4) and (5) we observe that  $10 \leq m < 38$  if  $5 \leq n \leq 17$ , since  $B(2^h, n)$  is non-decreasing (in  $n$ ) and  $B(2^h, 17) = 38$ . If  $10 \leq m < 38$  then  $m = 25$  in case (4),  $m = 18$  in case (5). But  $D(2^h, n) \neq 25$  if  $n \geq 4$  since  $3^4 \equiv 4^4 \pmod{25}$ ,  $D(2^h, n) \neq 18$  if  $n \geq 9$  since  $3^2 \equiv 9^2 \pmod{18}$ ,  $D(2^h, n) \neq 18$  if  $n \leq 7$  by Lemma 3.

**Proof of the Theorem.** By Lemmas 2 and 3 we have  $D(2^h, n) \leq B(2^h, n)$  for  $n \neq 1, 2, 4, 8$ . By Lemmas 4 and 5,  $D(2^h, n) \geq B(2^h, n)$  for  $5 \leq n \leq 7$  and  $n \geq 9$ . This proves the theorem for  $n \geq 5, n \neq 8$ .

Trivially,  $D(2^h, 1) = 1$  and  $D(2^h, 2) = 2; D(2^h, 3) = 6$  by Lemmas 3 and 4. It remains to determine  $D(2^h, 4)$  and  $D(2^h, 8)$ .

By Lemma 4,  $D(2^h, 8) \geq 16; D(2^h, 8) \geq 18$  since  $4^2 \equiv 8^2 \pmod{16}$  and  $1^4 \equiv 4^4 \pmod{17}$ . In fact,  $D(2^h, 8) = 18$ . Indeed, if (3.1) holds for  $m = 18$ , then (3.3) does also, hence  $a^2 \equiv b^2 \pmod{9}$  and  $a \equiv b \pmod{2}$ . But if  $a^2 \equiv b^2 \pmod{9}$  and  $1 \leq a < b \leq 8$  then  $a + b = 9, a \not\equiv b \pmod{2}$ .

For  $D(2^h, 4)$ , Lemma 4 yields  $D(2^h, 4) \geq 8$ ; and  $D(2^h, 4) \neq 8$  since  $2^4 \equiv 4^4 \pmod{8}$ . An argument similar to the one used for  $D(2^h, 8)$  will show that  $D(2^h, 4) = 9$  (start from (3.1) with  $m = 9$ ).

**4. Other results**

The following table gives the values of  $D(j, n)$  for  $2 \leq j \leq 30, 1 \leq n \leq 20$ .

$j \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	1	2	6	9	10	13	14	17	19	22	22	26	26	29	31	34	34	37	38	41
3	1	2	3	5	5	6	10	10	10	10	11	15	15	15	15	17	17	22	22	22
4	1	2	6	9	11	14	14	18	19	22	22	31	31	31	31	38	38	38	38	43
5	1	2	3	5	5	6	7	10	10	10	10	13	13	13	14	15	17	17	19	19
6	1	2	6	10	10	17	17	17	22	22	22	29	29	29	34	34	34	41	41	41
7	1	2	3	5	5	6	7	10	10	10	11	13	13	14	15	17	17	19	19	21
8	1	2	6	9	11	14	14	18	19	22	22	31	31	31	31	38	38	38	38	43
9	1	2	3	5	5	6	10	10	10	10	11	15	15	15	15	17	17	22	22	22
10	1	2	6	9	10	13	14	17	19	23	23	26	26	29	34	34	34	37	38	43
11	1	2	3	5	5	6	7	10	10	10	11	13	13	14	15	17	17	19	19	21
12	1	2	6	11	11	22	22	22	22	22	22	46	46	46	46	46	46	46	46	46
13	1	2	3	5	5	6	7	10	10	10	11	13	13	14	15	17	17	19	19	21
14	1	2	6	9	10	13	14	17	19	22	22	26	26	31	31	34	34	37	38	41
15	1	2	3	5	5	6	10	10	10	10	15	15	15	15	15	17	17	23	23	23
16	1	2	6	9	11	14	14	18	19	22	22	31	31	31	31	38	38	38	38	43
17	1	2	3	5	5	6	7	10	10	10	11	13	13	14	15	17	17	19	19	21
18	1	2	6	10	10	17	17	17	22	22	22	29	29	29	34	34	34	41	41	41
19	1	2	3	5	5	6	7	10	10	10	11	13	13	14	15	17	17	19	19	21
20	1	2	6	9	14	14	14	18	19	23	23	38	38	38	38	38	38	38	38	43
21	1	2	3	5	5	6	10	10	10	10	11	15	15	15	15	17	17	22	22	22
22	1	2	6	9	10	13	14	17	19	22	22	26	26	29	31	34	34	37	38	41
23	1	2	3	5	5	6	7	10	10	10	11	13	13	14	15	17	17	19	19	21
24	1	2	6	11	11	22	22	22	22	22	22	46	46	46	46	46	46	46	46	46
25	1	2	3	5	5	6	7	10	10	10	13	13	13	14	15	17	17	19	19	21
26	1	2	6	9	10	13	14	17	19	22	22	26	26	29	31	34	34	37	38	41
27	1	2	3	5	5	6	10	10	10	10	11	15	15	15	15	17	17	22	22	22
28	1	2	6	9	11	14	14	18	19	22	22	31	31	31	31	38	38	38	38	46
29	1	2	3	5	5	6	7	10	10	10	11	13	13	14	15	17	17	19	19	21
30	1	2	6	10	10	17	17	17	23	23	23	29	29	29	34	34	34	46	46	46

Further computer assisted calculations have shown that  $D(100, 100) = 206$ ,  $D(600, 600) = 1223$ ,  $D(1000, 1000) = 2003$  and  $D(2000, 2000) = 4003$ .

The following results are proved in [2] for sufficiently large  $n$ .

Let  $p_1, \dots, p_r$  be odd primes and  $a_1, \dots, a_r$  positive integers.

1. If  $j = p_1^{a_1} \dots p_r^{a_r}$ , then

$$D(j, n) = \min \{k | k \geq n, k \text{ squarefree and not divisible by any } p \equiv 1 \pmod{p_i}, i = 1, \dots, r\}.$$

2. If  $j = 2p_1^{a_1} \dots p_r^{a_r}$ , then

$$D(j, n) = \min \{k | k \geq 2n, k = p \text{ or } 2p \text{ with } p \not\equiv 1 \pmod{p_i}, i = 1, \dots, r\}.$$

3. If  $j = 2^a p_1^{a_1} \dots p_r^{a_r}$  with  $a \geq 2$ , then

$$D(j, n) = \min \{k \mid k \geq 2n, k = p \text{ or } 2p \text{ with } p \not\equiv 1 \pmod{p_i}, i = 1, \dots, r, \\ \text{and } p \equiv 3 \pmod{4}\}.$$

In [6] a proof is given for  $j = 3$  and  $j = 6$ , and all  $n$ , using the theory of binary quadratic forms.

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## Un problème de probabilité maximale

Dans un récent article [1], on trouve le théorème suivant:

**Théorème:** Si  $X_1, \dots, X_n$  sont des variables aléatoires indépendantes, distribuées selon des lois géométriques de paramètres  $r_1, \dots, r_n$ , le maximum de la probabilité  $P(X_1 + X_2 + \dots + X_n = i)$  est atteint, pour  $n$  et  $i$  donnés, lorsque  $r_1 = \dots = r_n = i/(n + i)$ .

Nous donnons ici une démonstration élémentaire du théorème. Soit  $F$  l'ensemble des lois de probabilités sur  $N$  avec la convolution  $p * q(i) = \sum p(j)q(i - j)$ . Soit  $g_r(i) = s \cdot r^i$  (avec  $s = 1 - r$ ) la loi géométrique de paramètre  $r \geq 0$  et  $G_n = \{g_{r(1)} * g_{r(2)} * \dots * g_{r(n)}\}$  l'ensemble des convolutions de  $n$  lois géométriques.

**Lemme 1:** Si  $p \in G_n$  et que l'on définit  $\Delta p(i) = p(i) - p(i - 1)$ , il existe  $i_0 = i_0(p)$  tel que  $\Delta p(i) \geq 0$  pour  $i < i_0$  et  $\Delta p(i) < 0$  pour  $i \geq i_0$  (unimodularité).