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## A new geometric inequality

Let  $\omega \in (0, \pi)$  be defined by the equation

$$\cot \omega = \cot \alpha_1 + \cot \alpha_2 + \cot \alpha_3 \tag{1}$$

where  $\alpha_1, \alpha_2, \alpha_3$ , are positive numbers satisfying

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi . \tag{2}$$

If  $\alpha_1, \alpha_2$  and  $\alpha_3$  are interpreted as the three angles of a triangle ( $T$ ), then  $\omega$  is called the Brocard angle of ( $T$ ) and there exists a number of identities relating  $\omega$  and  $\alpha_1, \alpha_2$  and  $\alpha_3$  [4]. This note is concerned with the problem of finding inequalities between  $\omega$  and  $\alpha_1, \alpha_2$  and  $\alpha_3$ . Since the appearance of [1], this problem has received much attention. At present the following inequalities are known [1–3].

$$2 \omega \leq \frac{1}{3} (\alpha_1 + \alpha_2 + \alpha_3) = \frac{\pi}{3} . \tag{3}$$

This is the oldest known inequality and follows from the inequality  $\cot^2 \omega \geq 3$  which is readily obtained from (1). The next inequality is

$$2 \omega \leq \sqrt[3]{\alpha_1 \alpha_2 \alpha_3} \tag{4}$$

which was proved in [1]. It is sharper than (3).

In [2] it was shown that

$$\omega^3 \leq (\alpha_1 - \omega) (\alpha_2 - \omega) (\alpha_3 - \omega) , \tag{5}$$

an inequality that implies (4).

Using the method of Lagrange multipliers, Mascioni [5] proved the inequality

$$2 \omega \leq 3 \left( \sum 1/\alpha_i \right)^{-1} . \tag{6}$$

This inequality is sharper than (4), since the harmonic mean of three numbers is less than or equal to their geometric mean. A different proof of (6) appears in [3]. Since

(5) implies that  $\omega$  is less than or equal to the geometric mean of  $\alpha_1 - \omega$ ,  $\alpha_2 - \omega$  and  $\alpha_3 - \omega$ , it is natural to ask how  $\omega$  is related to their harmonic mean. In the present note we prove

**Theorem 1.** *If  $\omega$  is defined by (1) and (2) then*

$$\omega \geq 3 \left\{ \sum 1/(\alpha_i - \omega) \right\}^{-1}. \quad (7)$$

*Equality holds if and only if  $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$ .*

**Proof:** Let  $f(x) = \cot x - \frac{1}{x}$  where  $0 < x < \pi$ .

Then  $f'(x) = -\frac{1}{\sin^2 x} + \frac{1}{x^2}$  and  $f''(x) = 2 \csc^3 x \left\{ \cos x - \left( \frac{\sin x}{x} \right)^3 \right\}$ . Since  $\sin x < x$

for  $0 < x < \pi$  we have  $f'(x) < 0$ . In [5], it was shown that  $\left( \frac{\sin x}{x} \right)^3 > \cos x$  in  $(0, \pi)$ .

Then  $f''(x) < 0$  in  $(0, \pi)$ . It follows that  $f$  is a concave decreasing function in  $(0, \pi)$ . In particular, if  $x_1, x_2, \dots, x_6$  are six numbers in  $(0, \pi)$ , then

$$f\left(\frac{x_1 + x_2 + \dots + x_6}{6}\right) \geq \frac{1}{6} \{f(x_1) + f(x_2) + \dots + f(x_6)\}.$$

If this inequality is used with the six numbers  $\alpha_1 - \omega, \alpha_2 - \omega, \alpha_3 - \omega, \omega, \omega, \omega$ , all of which lie in  $(0, \pi)$  we obtain

$$\begin{aligned} \sum_{i=1}^3 \left\{ \cot(\alpha_i - \omega) - \frac{1}{\alpha_i - \omega} + \cot \omega - \frac{1}{\omega} \right\} &\leq 6 \left\{ \cot(\pi/6) - (6/\pi) \right\} \\ &\leq 6 \left\{ \cot \omega - \frac{1}{\omega} \right\}, \end{aligned} \quad (8)$$

where the last inequality follows because  $\omega \leq \pi/6$  and  $\cot x - 1/x$  is decreasing in  $(0, \pi)$ .

From (8) we obtain

$$\sum_{i=1}^3 \{ \cot(\alpha_i - \omega) - \cot \omega \} + 3/\omega \leq \sum_{i=1}^3 \frac{1}{\alpha_i - \omega}. \quad (9)$$

The proof of (7) will be complete if we show that the sum on the left-hand side of (9) is positive. This is the difficult part of the proof. It depends, in part, on the following identities satisfied by  $\omega$ . Their derivation is quite easy.

$$(i) \quad \cot \alpha_1 \cot \alpha_2 + \cot \alpha_2 \cot \alpha_3 + \cot \alpha_3 \cot \alpha_1 = 1;$$

$$(ii) \quad \prod_{i=1}^3 (\cot \omega - \cot \alpha_i) = \prod_{i=1}^3 \csc \alpha_i;$$

$$(iii) \sum_{i=1}^3 \csc^2 \alpha_i = \csc^2 \omega ; \tag{10}$$

$$(iv) \sum_{i=1}^3 \csc^4 \alpha_i + 4 \cot \omega \prod_{i=1}^3 \csc \alpha_i = \csc^4 \omega .$$

We now consider the sum on the left-hand side of (9). We have

$$\begin{aligned} \cot(\alpha_i - \omega) - \cot \omega &= \frac{\cot \alpha_i \cot \omega + 1}{\cot \omega - \cot \alpha_i} - \cot \omega = \frac{-\cot^2 \omega + 2 \cot \omega \cot \alpha_i + 1}{\cot \omega - \cot \alpha_i} \\ &= -(\cot \omega - \cot \alpha_i) + \frac{\csc^2 \alpha_i}{\cot \omega - \cot \alpha_i} . \end{aligned}$$

Thus

$$\sum_{i=1}^3 \{\cot(\alpha_i - \omega) - \cot \omega\} = -2 \cot \omega + \sum_{i=1}^3 \frac{\csc^2 \alpha_i}{\cot \omega - \cot \alpha_i} . \tag{11}$$

From (10; (i)) we have

$$\begin{aligned} \cot \alpha_1 \cot \alpha_2 &= 1 - \cot \alpha_3 (\cot \alpha_1 + \cot \alpha_2) = 1 - \cot \alpha_3 (\cot \omega - \cot \alpha_3) \\ &= \csc^2 \alpha_3 - \cot \alpha_3 \cot \omega . \end{aligned}$$

Thus

$$\begin{aligned} &(\cot \omega - \cot \alpha_1) (\cot \omega - \cot \alpha_2) \\ &= \cot^2 \omega - \cot \omega (\cot \alpha_1 + \cot \alpha_2) + \csc^2 \alpha_3 - \cot \alpha_3 \cot \omega = \csc^2 \alpha_3 . \end{aligned} \tag{12}$$

A similar formula holds for  $\csc^2 \alpha_1$  and  $\csc^2 \alpha_2$ . Returning to the second sum in (11) and using ((10); (ii)) and (12) we obtain

$$\sum_{i=1}^3 \frac{\csc^2 \alpha_i}{\cot \omega - \cot \alpha_i} = \frac{1}{\prod_{i=1}^3 (\cot \omega - \cot \alpha_i)} \sum_{i=1}^3 \csc^4 \alpha_i = \frac{1}{\prod_{i=1}^3 \csc \alpha_i} \sum_{i=1}^3 \csc^4 \alpha_i . \tag{13}$$

If we use (13) in (11) and then use ((10); (iv)) we obtain

$$\begin{aligned} \sum_{i=1}^3 \{\cot(\alpha_i - \omega) - \cot \omega\} &= \frac{1}{\prod_{i=1}^3 \csc \alpha_i} \left\{ \sum_{i=1}^3 \csc^4 \alpha_i - 2 \cot \omega \prod_{i=1}^3 \csc \alpha_i \right\} \\ &= \frac{1}{\prod_{i=1}^3 \csc \alpha_i} \left\{ \sum_{i=1}^3 \csc^4 \alpha_i + \frac{1}{2} \sum_{i=1}^3 \csc^4 \alpha_i - \frac{1}{2} \csc^4 \omega \right\} = \frac{1}{2 \prod_{i=1}^3 \csc \alpha_i} \left\{ 3 \sum_{i=1}^3 \csc^4 \alpha_i - \csc^4 \omega \right\} . \end{aligned} \tag{14}$$

Now, from ((10); (iii)) we have

$$\{\csc^2 \omega\}^2 = \left\{ \sum_{i=1}^3 \csc^2 \alpha_i \right\}^2 \leq 3 \sum_{i=1}^3 \csc^4 \alpha_i. \quad (15)$$

Thus the right-hand side of (14) is positive; it is zero if and only if  $\alpha_1 = \alpha_2 = \alpha_3$ . It follows that the right-hand in (9) is greater than or equal to  $3/\omega$  with equality if and only if  $\alpha_1 = \alpha_2 = \alpha_3$ . This finishes the proof of Theorem 1.

We end this note by remarking that a straightforward application of Holder's inequality on (7) gives

$$\frac{3}{\omega^\lambda} \leq \sum_{i=1}^3 \frac{1}{(\alpha_i - \omega)^\lambda} \quad (16)$$

for every  $\lambda \geq 1$ .

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## On some inequalities connected with Fermat's equation

### 1. Introduction

In 1856 I. A. Grünert ([3], see also [6] p. 226) proved that if  $n$  is an integer,  $n \geq 2$  and  $0 < x < y < z$  are real numbers satisfying the equation

$$x^n + y^n = z^n \quad (1)$$

then

$$z - y < \frac{x}{n}. \quad (2)$$

This result was rediscovered by G. Towses [7], and then by D. Zeitlin [8].