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# Generalization of a formula of C. Buchta about the convex hull of random points

## 1. Introduction

For an arbitrary convex body  $K$  in the  $d$ -dimensional Euclidean space  $E^d (d \geq 2)$  we denote by  $V_n^{(d)}(K)$  the expected volume of the convex hull  $H_n$  of  $n$  random points chosen independently and uniformly inside  $K$ . The special case  $V_3^{(2)}(K)$  is directly related to a classical problem of geometrical probability, known as «Sylvester's problem»:

What is the probability  $p(K)$  that 4 points chosen identically and uniformly at random from the interior of a plane convex set  $K$  form a convex quadrilateral?

It is easy to see [6], pp. 63–64, that

$$p(K) = 1 - \frac{4}{F(K)} V_3^{(2)}(K), \tag{1.1}$$

where  $F(K)$  is the area of  $K$ .

For further results and a complete list of references concerning the convex hull of random points and Sylvester's problem, see the recent survey of Buchta [2].

For arbitrary plane convex sets, respectively three-dimensional convex bodies, Buchta [1] proves the following relationships

$$V_4^{(2)}(K) = 2 V_3^{(2)}(K) \tag{1.2}$$

and

$$V_5^{(3)}(K) = \frac{5}{2} V_4^{(3)}(K). \tag{1.3}$$

The aim of this note is to generalize Buchta's formulas (1.2) and (1.3) in the following sense:

**Theorem 1.** Let  $K$  be an arbitrary plane convex set. Then

$$V_{2m}^{(2)}(K) = \sum_{k=1}^{m-1} \alpha_{2m-2k+1} V_{2m-2k+1}^{(2)}(K) \quad m = 2, 3, \dots, \tag{1.4}$$

where  $\alpha_{2m-2k+1}$  are constants defined by the recursion formula

$$\alpha_{2m-1} = m, \tag{1.4'}$$

$$\alpha_{2m-2k+1} = \frac{m}{2m-2k+1} \left[ \binom{2m-1}{2m-2k} - \sum_{i=1}^{k-1} \frac{2m-2i+1}{2m} \binom{2m-2i}{2m-2k} \alpha_{2m-2i+1} \right],$$

$$\text{for } k = 2, 3, \dots, m-1. \tag{1.4''}$$

**Theorem 2.** Let  $K$  be an arbitrary three-dimensional convex body. Then

$$V_{2m+1}^{(3)}(K) = \sum_{k=1}^{m-1} \beta_{2m-2k+2} V_{2m-2k+2}^{(3)}(K) \quad m = 2, 3, \dots, \quad (1.5)$$

where  $\beta_{2m-2k+2}$  are constants defined by the recursion formula

$$\beta_{2m} = \frac{2m+1}{2}, \quad (1.5')$$

$$\beta_{2m-2k+2} = \frac{m(2m+1)}{(2m-2k+1)(2m-2k+2)} \cdot \left[ \binom{2m-1}{2m-2k} - \sum_{i=1}^{k-1} \frac{(2m-2i+1)(2m-2i+2)}{2m(2m+1)} \binom{2m-2i}{2m-2k} \beta_{2m-2i+2} \right]$$

$$\text{for } k = 2, 3, \dots, m-1. \quad (1.5'')$$

## 2. Proof of Theorem 1

We can assume that  $F(K) = 1$  since, under an affine transformation, the expected area  $V_n^{(2)}(K)$  changes only through  $F(K)$ .

Rényi and Sulanke [5], p. 76, show that

$$E_{n+1}^{(2)}(K) = \binom{n+1}{2} \int_K \int_K [\tilde{F}^{n-1} + (1 - \tilde{F})^{n-1}] dP_1 dP_2, \quad (2.1)$$

where  $E_{n+1}^{(2)}(K)$  is the expected number of vertices of the convex hull  $H_{n+1}$  of  $n+1$  random points  $P_1, P_2, \dots, P_{n+1}$  chosen independently and uniformly inside  $K$ , and  $\tilde{F} = \tilde{F}(P_1, P_2)$  denotes the area of the smaller of the two parts of  $K$  cut off by the line through  $P_1$  and  $P_2$ .

Efron [4], p. 335, relates the expected volume  $V_n^{(d)}(K)$  of the convex hull  $H_n$  to the expected number of vertices  $E_{n+1}^{(d)}(K)$  of the convex hull  $H_{n+1}$  by the following formula

$$V_n^{(d)}(K) = 1 - \frac{1}{n+1} E_{n+1}^{(d)}(K). \quad (2.2)$$

Using (2.2) expression (2.1) becomes

$$V_n^{(2)}(K) = 1 - \frac{n}{2} \int_K \int_K [\tilde{F}^{n-1} + (1 - \tilde{F})^{n-1}] dP_1 dP_2. \quad (2.3)$$

Consider now our formula (1.4)

$$V_{2m}^{(2)}(K) = \alpha_{2m-1} V_{2m-1}^{(2)}(K) + \dots + \alpha_{2m-2k+1} V_{2m-2k+1}^{(2)}(K) + \dots + \alpha_3 V_3^{(2)}(K). \quad (2.4)$$

Taking into account (2.3) we obtain

$$\begin{aligned}
 & 1 - m \int_K \int_K [\tilde{F}^{2m-1} + (1 - \tilde{F})^{2m-1}] dP_1 dP_2 = \\
 & = \alpha_{2m-1} \left( 1 - \frac{2m-1}{2} \int_K \int_K [\tilde{F}^{2m-2} + (1 - \tilde{F})^{2m-2}] dP_1 dP_2 \right) + \\
 & \quad + \dots + \alpha_{2m-2k+1} \left( 1 - \frac{2m-2k+1}{2} \int_K \int_K [\tilde{F}^{2m-2k} + (1 - \tilde{F})^{2m-2k}] dP_1 dP_2 \right) + \\
 & \quad + \dots + \alpha_3 \left( 1 - \frac{3}{2} \int_K \int_K [\tilde{F}^2 + (1 - \tilde{F})^2] dP_1 dP_2 \right), \tag{2.5}
 \end{aligned}$$

or, by developing the integrands,

$$\begin{aligned}
 & 1 - m \int_K \int_K \left[ \sum_{i=0}^{2m-2} \binom{2m-1}{i} (-1)^i \tilde{F}^i \right] dP_1 dP_2 = \\
 & = \alpha_{2m-1} \left( 1 - (2m-1) \int_K \int_K \tilde{F}^{2m-2} dP_1 dP_2 - \right. \\
 & \quad \left. - \frac{2m-1}{2} \int_K \int_K \left[ \sum_{i=0}^{2m-3} \binom{2m-2}{i} (-1)^i \tilde{F}^i \right] dP_1 dP_2 \right) + \\
 & \quad + \dots + \alpha_{2m-2k+1} \left( 1 - (2m-2k+1) \int_K \int_K \tilde{F}^{2m-2k} dP_1 dP_2 - \right. \\
 & \quad \left. - \frac{2m-2k+1}{2} \int_K \int_K \left[ \sum_{i=0}^{2m-2k-1} \binom{2m-2k}{i} (-1)^i \tilde{F}^i \right] dP_1 dP_2 \right) + \\
 & \quad + \dots + \alpha_3 \left( 1 - 3 \int_K \int_K \tilde{F}^2 dP_1 dP_2 - \frac{3}{2} \int_K \int_K [-2\tilde{F} + 1] dP_1 dP_2 \right). \tag{2.6}
 \end{aligned}$$

Comparing the coefficients of  $\int_K \int_K \tilde{F}^i dP_1 dP_2$  ( $i = 0, 1, \dots, 2m-2$ ) in (2.6) we get

$$m \binom{2m-1}{2m-2} = (2m-1) \alpha_{2m-1}, \tag{2.7}$$

$$m \binom{2m-1}{2m-2k} = \frac{2m-1}{2} \binom{2m-2}{2m-2k} \alpha_{2m-1} + \dots + (2m-2k+1) \alpha_{2m-2k+1}, \tag{2.8}$$

$2 \leq k \leq m-1,$

$$\begin{aligned}
 m \binom{2m-1}{2m-2k-1} &= \frac{2m-1}{2} \binom{2m-2}{2m-2k-1} \alpha_{2m-1} + \\
 & \quad + \dots + \frac{2m-2k+1}{2} \binom{2m-2k}{2m-2k-1} \alpha_{2m-2k+1}, \quad 1 \leq k \leq m-1 \tag{2.9}
 \end{aligned}$$

and

$$m - 1 = \frac{2m - 3}{2} \alpha_{2m-1} + \dots + \frac{2m - 2k + 1}{2} \alpha_{2m-2k+3} + \dots + \frac{1}{2} \alpha_3. \quad (2.10)$$

From (2.7) and (2.8) we easily conclude that

$$\alpha_{2m-1} = m \quad (2.11)$$

and

$$\alpha_{2m-2k+1} = \frac{m}{2m-2k+1} \left[ \binom{2m-1}{2m-2k} - \sum_{i=1}^{k-1} \frac{2m-2i+1}{2m} \binom{2m-2i}{2m-2k} \alpha_{2m-2i+1} \right], \quad 2 \leq k \leq m-1, \quad (2.12)$$

in accordance with (1.4') and (1.4'').

Finally, expressions (2.9) and (2.10) can be verified by an adequate addition of the constants defined by (2.12).

### 3. Proof of Theorem 2

Analogously to the proof of Theorem 1 we can assume the volume  $V(K)$  of the convex body  $K$  to be 1 since, under an affine transformation, the expected volume  $V_n^{(3)}(K)$  changes only through  $V(K)$ .

Rényi and Sulanke's integral expression (2.1) can be extended to higher dimensions. For the three-dimensional case we can state

$$F_{n+1}^{(3)}(K) = \binom{n+1}{3} \int_K \int_K \int_K [\tilde{V}^{n-2} + (1 - \tilde{V})^{n-2}] dP_1 dP_2 dP_3, \quad (3.1)$$

where  $F_{n+1}^{(3)}(K)$  is the expected number of faces of the convex hull  $H_{n+1}$  and  $\tilde{V} = \tilde{V}(P_1, P_2, P_3)$  denotes the volume of the smaller of the two parts of  $K$  cut off by the plane through  $P_1, P_2$  and  $P_3$  (e.g. Buchta [1], p. 155).

With probability 1 all faces of  $H_{n+1}$  will be triangles. Therefore, taking into account expression (3.1), Efron's formula (2.2) and Euler's theorem, we conclude

$$V_n^{(3)}(K) = 1 - \frac{2}{n+1} - \frac{(n-1)n}{12} \int_K \int_K \int_K [\tilde{V}^{n-2} + (1 - \tilde{V})^{n-2}] dP_1 dP_2 dP_3. \quad (3.2)$$

To prove Theorem 2 we use the same procedure as above. We now consider formula (1.5)

$$V_{2m+1}^{(3)}(K) = \beta_{2m} V_{2m}^{(3)}(K) + \dots + \beta_{2m-2k+2} V_{2m-2k+2}^{(3)}(K) + \dots + \beta_4 V_4^{(3)}(K). \quad (3.3)$$

Taking into account expression (3.2) we obtain

$$\begin{aligned}
 & 1 - \frac{2}{2m+2} - \frac{2m(2m+1)}{12} \int_K \int_K \int_K [\tilde{V}^{2m-1} + (1-\tilde{V})^{2m-1}] dP_1 dP_2 dP_3 = \\
 & = \beta_{2m} \left( 1 - \frac{2}{2m+1} - \frac{(2m-1)2m}{12} \int_K \int_K \int_K [\tilde{V}^{2m-2} + (1-\tilde{V})^{2m-2}] dP_1 dP_2 dP_3 \right) + \\
 & \quad + \dots + \beta_{2m-2k+2} \left( 1 - \frac{2}{2m-2k+3} - \frac{(2m-2k+1)(2m-2k+2)}{12} \right. \\
 & \quad \quad \cdot \int_K \int_K \int_K [\tilde{V}^{2m-2k} + (1-\tilde{V})^{2m-2k}] dP_1 dP_2 dP_3 \left. \right) + \\
 & \quad + \dots + \beta_4 \left( 1 - \frac{2}{5} - \frac{3 \cdot 4}{12} \int_K \int_K \int_K [\tilde{V}^2 + (1-\tilde{V})^2] dP_1 dP_2 dP_3 \right), \tag{3.4}
 \end{aligned}$$

or, by developing the integrants,

$$\begin{aligned}
 & 1 - \frac{2}{2m+2} - \frac{2m(2m+1)}{12} \int_K \int_K \int_K \left[ \sum_{i=0}^{2m-2} \binom{2m-1}{i} (-1)^i \tilde{V}^i \right] dP_1 dP_2 dP_3 = \\
 & = \beta_{2m} \left( 1 - \frac{2}{2m+1} - \frac{2(2m-1)2m}{12} \int_K \int_K \int_K \tilde{V}^{2m-2} dP_1 dP_2 dP_3 - \right. \\
 & \quad \left. - \frac{(2m-1)2m}{12} \int_K \int_K \int_K \left[ \sum_{i=0}^{2m-3} \binom{2m-2}{i} (-1)^i \tilde{V}^i \right] dP_1 dP_2 dP_3 \right) + \\
 & \quad + \dots + \beta_{2m-2k+2} \left( 1 - \frac{2}{2m-2k+3} - \frac{2(2m-2k+1)(2m-2k+2)}{12} \right. \\
 & \quad \quad \cdot \int_K \int_K \int_K \tilde{V}^{2m-2k} dP_1 dP_2 dP_3 - \frac{(2m-2k+1)(2m-2k+2)}{12} \cdot \\
 & \quad \quad \cdot \int_K \int_K \int_K \left[ \sum_{i=0}^{2m-2k-1} \binom{2m-2k}{i} (-1)^i \tilde{V}^i \right] dP_1 dP_2 dP_3 \left. \right) + \\
 & \quad + \dots + \beta_4 \left( 1 - \frac{2}{5} - 2 \int_K \int_K \int_K \tilde{V}^2 dP_1 dP_2 dP_3 - \int_K \int_K \int_K [-2\tilde{V} + 1] dP_1 dP_2 dP_3 \right). \tag{3.5}
 \end{aligned}$$

Comparing the coefficients of  $\int_K \int_K \int_K \tilde{V}^i dP_1 dP_2 dP_3$  ( $i = 0, 1, \dots, 2m-2$ ) in (3.5) we get

$$\frac{2m(2m+1)}{12} \binom{2m-1}{2m-2} = \frac{2(2m-1)2m}{12} \beta_{2m}, \tag{3.6}$$

$$\begin{aligned}
 & \frac{2m(2m+1)}{12} \binom{2m-1}{2m-2k} = \frac{(2m-1)2m}{12} \binom{2m-2}{2m-2k} \beta_{2m} + \\
 & + \dots + \frac{2(2m-2k+1)(2m-2k+2)}{12} \beta_{2m-2k+2} \quad \text{for } k = 2, 3, \dots, m-1, \tag{3.7}
 \end{aligned}$$

$$\begin{aligned} \frac{2m(2m+1)}{12} \binom{2m-1}{2m-2k-1} &= \frac{(2m-1)2m}{12} \binom{2m-2}{2m-2k-1} \beta_{2m} + \\ + \dots + \frac{(2m-2k+1)(2m-2k+2)}{12} \binom{2m-2k}{2m-2k-1} \beta_{2m-2k+2} & \end{aligned}$$

for  $k = 1, 2, \dots, m-1$  (3.8)

and

$$\begin{aligned} 1 - \frac{2}{2m+2} - \frac{2m(2m+1)}{12} &= \left( 1 - \frac{2}{2m+1} - \frac{(2m-1)2m}{12} \right) \beta_{2m} + \\ + \dots + \left( 1 - \frac{2}{2m-2k+3} - \frac{(2m-2k+1)(2m-2k+2)}{12} \right) \beta_{2m-2k+2} &+ \\ + \dots + \left( 1 - \frac{2}{5} - 1 \right) \beta_4. & \end{aligned}$$

(3.9)

From (3.6) and (3.7) we easily conclude that

$$\beta_{2m} = \frac{2m+1}{2} \quad (3.10)$$

and

$$\begin{aligned} \beta_{2m-2k+2} &= \frac{m(2m+1)}{(2m-2k+1)(2m-2k+2)} \cdot \\ \cdot \left[ \binom{2m-1}{2m-2k} - \sum_{i=1}^{k-1} \frac{(2m-2i+1)(2m-2i+2)}{2m(2m+1)} \binom{2m-2i}{2m-2k} \beta_{2m-2i+2} \right] & \end{aligned}$$

for  $k = 2, 3, \dots, m-1$ , (3.11)

in accordance with (1.5') and (1.5'').

Finally, expressions (3.8) and (3.9) can be verified by an adequate addition of the constants defined by (3.11).

#### 4. Some remarks

**Remark 1.** For  $m = 2$  expressions (1.4) and (1.5) become

$$V_4^{(2)}(K) = 2V_3^{(2)}(K) \quad \text{and} \quad V_5^{(3)}(K) = \frac{5}{2}V_4^{(3)}(K),$$

coincident with Buchta's formulas (1.2) and (1.3).

**Remark 2.** Some numerical values for the planar case:

$$V_4^{(2)}(K) = 2 V_3^{(2)}(K),$$

$$V_6^{(2)}(K) = 3 V_5^{(2)}(K) - 5 V_3^{(2)}(K),$$

$$V_8^{(2)}(K) = 4 V_7^{(2)}(K) - 14 V_5^{(2)}(K) + 28 V_3^{(2)}(K),$$

$$V_{10}^{(2)}(K) = 5 V_9^{(2)}(K) - 30 V_7^{(2)}(K) + 126 V_5^{(2)}(K) - 255 V_3^{(2)}(K).$$

**Remark 3.** Some numerical values for the three-dimensional case:

$$V_5^{(3)}(K) = \frac{5}{2} V_4^{(3)}(K),$$

$$V_7^{(3)}(K) = \frac{7}{2} V_6^{(3)}(K) - \frac{35}{4} V_4^{(3)}(K),$$

$$V_9^{(3)}(K) = \frac{9}{2} V_8^{(3)}(K) - 21 V_6^{(3)}(K) + 63 V_4^{(3)}(K).$$

**Remark 4.** Recently, again Buchta [3], p. 96, generalizes his results (1.2) and (1.3) to higher dimensions. For an arbitrary  $d$ -dimensional convex body he shows that

$$V_{d+2}^{(d)}(K) = \frac{d+2}{2} V_{d+1}^{(d)}(K).$$

**Remark 5.** Unfortunately, the procedure to relate the expected volume  $V_n^{(d)}(K)$ ,  $d = 2, 3$ , of the convex hull of  $n$  random points to the integral expressions (2.3) and (3.2) cannot be extended to higher dimensions. Therefore it seems to be difficult to generalize our results to higher dimensions.

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