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Autor: Affentranger, Fernando

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Generalization of a formula of C. Buchta about the convex hull of random points

1. Introduction

For an arbitrary convex body K in the d-dimensional Euclidean space $E^d (d \geqq 2)$ we denote by $V_n^{(d)}(K)$ the expected volume of the convex hull H_n of n random points chosen independently and uniformly inside K. The special case $V_3^{(2)}(K)$ is directly related to a classical problem of geometrical probability, known as «Sylvester's problem»:

> What is the probability $p(K)$ that 4 points chosen identically and uniformly at random from the interior of a plane convex set K form a convex quadrilateral?

It is easy to see [6], pp. 63–64, that

$$p(K) = 1 - \frac{4}{F(K)} V_3^{(2)}(K), \tag{1.1}$$

where $F(K)$ is the area of K.
For further results and a complete list of references concerning the convex hull of random points and Sylvester's problem, see the recent survey of Buchta [2].
For arbitrary plane convex sets, respectively three-dimensional convex bodies, Buchta [1] proves the following relationships

$$V_4^{(2)}(K) = 2 V_3^{(2)}(K) \tag{1.2}$$

and

$$V_5^{(3)}(K) = \tfrac{5}{2} V_4^{(3)}(K). \tag{1.3}$$

The aim of this note is to generalize Buchta's formulas (1.2) and (1.3) in the following sense:

Theorem 1. Let K be an arbitrary plane convex set. Then

$$V_{2m}^{(2)}(K) = \sum_{k=1}^{m-1} \alpha_{2m-2k+1} V_{2m-2k+1}^{(2)}(K) \qquad m = 2, 3, \ldots, \tag{1.4}$$

where $\alpha_{2m-2k+1}$ are constants defined by the recursion formula

$$\alpha_{2m-1} = m, \tag{1.4'}$$

$$\alpha_{2m-2k+1} = \frac{m}{2m-2k+1}\left[\binom{2m-1}{2m-2k} - \sum_{i=1}^{k-1} \frac{2m-2i+1}{2m}\binom{2m-2i}{2m-2k}\alpha_{2m-2i+1}\right],$$

for $k = 2, 3, \ldots, m-1$. (1.4″)

Theorem 2. Let K be an arbitrary three-dimensional convex body. Then

$$V_{2m+1}^{(3)}(K) = \sum_{k=1}^{m-1} \beta_{2m-2k+2} V_{2m-2k+2}^{(3)}(K) \qquad m = 2, 3, \ldots, \tag{1.5}$$

where $\beta_{2m-2k+2}$ are constants defined by the recursion formula

$$\beta_{2m} = \frac{2m+1}{2}, \tag{1.5'}$$

$$\beta_{2m-2k+2} = \frac{m(2m+1)}{(2m-2k+1)(2m-2k+2)} \cdot \left[\binom{2m-1}{2m-2k} - \sum_{i=1}^{k-1} \frac{(2m-2i+1)(2m-2i+2)}{2m(2m+1)} \binom{2m-2i}{2m-2k} \beta_{2m-2i+2}\right]$$

for $k = 2, 3, \ldots, m-1$. (1.5″)

2. Proof of Theorem 1

We can assume that $F(K) = 1$ since, under an affine transformation, the expected area $V_n^{(2)}(K)$ changes only through $F(K)$.
Rényi and Sulanke [5], p. 76, show that

$$E_{n+1}^{(2)}(K) = \binom{n+1}{2} \int_K \int_K [\tilde{F}^{n-1} + (1-\tilde{F})^{n-1}]\, dP_1\, dP_2, \tag{2.1}$$

where $E_{n+1}^{(2)}(K)$ is the expected number of vertices of the convex hull H_{n+1} of $n+1$ random points $P_1, P_2, \ldots, P_{n+1}$ chosen independently and uniformly inside K, and $\tilde{F} = \tilde{F}(P_1, P_2)$ denotes the area of the smaller of the two parts of K cut off by the line through P_1 and P_2.
Efron [4], p. 335, relates the expected volume $V_n^{(d)}(K)$ of the convex hull H_n to the expected number of vertices $E_{n+1}^{(d)}(K)$ of the convex hull H_{n+1} by the following formula

$$V_n^{(d)}(K) = 1 - \frac{1}{n+1} E_{n+1}^{(d)}(K). \tag{2.2}$$

Using (2.2) expression (2.1) becomes

$$V_n^{(2)}(K) = 1 - \frac{n}{2} \int_K \int_K [\tilde{F}^{n-1} + (1-\tilde{F})^{n-1}]\, dP_1\, dP_2. \tag{2.3}$$

Consider now our formula (1.4)

$$V_{2m}^{(2)}(K) = \alpha_{2m-1} V_{2m-1}^{(2)}(K) + \ldots + \alpha_{2m-2k+1} V_{2m-2k+1}^{(2)}(K) + \ldots + \alpha_3 V_3^{(2)}(K). \tag{2.4}$$

Taking into account (2.3) we obtain

$$
\begin{aligned}
&1 - m \int_K \int_K [\tilde{F}^{2m-1} + (1-\tilde{F})^{2m-1}]\, dP_1\, dP_2 = \\
&= \alpha_{2m-1}\left(1 - \frac{2m-1}{2} \int_K \int_K [\tilde{F}^{2m-2} + (1-\tilde{F})^{2m-2}]\, dP_1\, dP_2\right) + \\
&\quad + \ldots + \alpha_{2m-2k+1}\left(1 - \frac{2m-2k+1}{2} \int_K \int_K [\tilde{F}^{2m-2k} + (1-\tilde{F})^{2m-2k}]\, dP_1\, dP_2\right) + \\
&\quad + \ldots + \alpha_3\left(1 - \frac{3}{2} \int_K \int_K [\tilde{F}^2 + (1-\tilde{F})^2]\, dP_1\, dP_2\right), \qquad (2.5)
\end{aligned}
$$

or, by developing the integrands,

$$
\begin{aligned}
&1 - m \int_K \int_K \left[\sum_{i=0}^{2m-2} \binom{2m-1}{i} (-1)^i \tilde{F}^i\right] dP_1\, dP_2 = \\
&= \alpha_{2m-1}\left(1 - (2m-1) \int_K \int_K \tilde{F}^{2m-2}\, dP_1\, dP_2 - \right. \\
&\qquad \left. - \frac{2m-1}{2} \int_K \int_K \left[\sum_{i=0}^{2m-3} \binom{2m-2}{i} (-1)^i \tilde{F}^i\right] dP_1\, dP_2\right) + \\
&\quad + \ldots + \alpha_{2m-2k+1}\left(1 - (2m-2k+1) \int_K \int_K \tilde{F}^{2m-2k}\, dP_1\, dP_2 - \right. \\
&\qquad \left. - \frac{2m-2k+1}{2} \int_K \int_K \left[\sum_{i=0}^{2m-2k-1} \binom{2m-2k}{i} (-1)^i \tilde{F}^i\right] dP_1\, dP_2\right) + \\
&\quad + \ldots + \alpha_3\left(1 - 3 \int_K \int_K \tilde{F}^2\, dP_1\, dP_2 - \frac{3}{2} \int_K \int_K [-2\tilde{F} + 1]\, dP_1\, dP_2\right). \qquad (2.6)
\end{aligned}
$$

Comparing the coefficients of $\int_K \int_K \tilde{F}^i\, dP_1\, dP_2$ $(i = 0, 1, \ldots, 2m-2)$ in (2.6) we get

$$
m \binom{2m-1}{2m-2} = (2m-1)\, \alpha_{2m-1}, \qquad (2.7)
$$

$$
m \binom{2m-1}{2m-2k} = \frac{2m-1}{2} \binom{2m-2}{2m-2k} \alpha_{2m-1} + \ldots + (2m-2k+1)\, \alpha_{2m-2k+1}, \quad 2 \leqq k \leqq m-1, \qquad (2.8)
$$

$$
\begin{aligned}
m \binom{2m-1}{2m-2k-1} &= \frac{2m-1}{2} \binom{2m-2}{2m-2k-1} \alpha_{2m-1} + \\
&\quad + \ldots + \frac{2m-2k+1}{2} \binom{2m-2k}{2m-2k-1} \alpha_{2m-2k+1}, \quad 1 \leqq k \leqq m-1 \qquad (2.9)
\end{aligned}
$$

and

$$m-1=\frac{2m-3}{2}\alpha_{2m-1}+\ldots+\frac{2m-2k+1}{2}\alpha_{2m-2k+3}+\ldots+\frac{1}{2}\alpha_3\,. \tag{2.10}$$

From (2.7) and (2.8) we easily conclude that

$$\alpha_{2m-1}=m \tag{2.11}$$

and

$$\alpha_{2m-2k+1}=\frac{m}{2m-2k+1}\left[\binom{2m-1}{2m-2k}-\sum_{i=1}^{k-1}\frac{2m-2i+1}{2m}\binom{2m-2i}{2m-2k}\alpha_{2m-2i+1}\right],$$
$$2\leqq k\leqq m-1\,, \tag{2.12}$$

in accordance with (1.4′) and (1.4″).
Finally, expressions (2.9) and (2.10) can be verified by an adequate addition of the constants defined by (2.12).

3. Proof of Theorem 2

Analogously to the proof of Theorem 1 we can assume the volume $V(K)$ of the convex body K to be 1 since, under an affine transformation, the expected volume $V_n^{(3)}(K)$ changes only through $V(K)$.
Rényi and Sulanke's integral expression (2.1) can be extended to higher dimensions. For the three-dimensional case we can state

$$F_{n+1}^{(3)}(K)=\binom{n+1}{3}\int_K\int_K\int_K[\tilde{V}^{n-2}+(1-\tilde{V})^{n-2}]\,dP_1\,dP_2\,dP_3\,, \tag{3.1}$$

where $F_{n+1}^{(3)}(K)$ is the expected number of faces of the convex hull H_{n+1} and $\tilde{V}=\tilde{V}(P_1,P_2,P_3)$ denotes the volume of the smaller of the two parts of K cut off by the plane through P_1, P_2 and P_3 (e.g. Buchta [1], p. 155).
With probability 1 all faces of H_{n+1} will be triangles. Therefore, taking into account expression (3.1), Efron's formula (2.2) and Euler's theorem, we conclude

$$V_n^{(3)}(K)=1-\frac{2}{n+1}-\frac{(n-1)\,n}{12}\int_K\int_K\int_K[\tilde{V}^{n-2}+(1-\tilde{V})^{n-2}]\,dP_1\,dP_2\,dP_3\,. \tag{3.2}$$

To prove Theorem 2 we use the same procedure as above. We now consider formula (1.5)

$$V_{2m+1}^{(3)}(K)=\beta_{2m}V_{2m}^{(3)}(K)+\ldots+\beta_{2m-2k+2}V_{2m-2k+2}^{(3)}(K)+\ldots+\beta_4V_4^{(3)}(K)\,. \tag{3.3}$$

Taking into account expression (3.2) we obtain

$$1-\frac{2}{2m+2}-\frac{2m(2m+1)}{12}\int_K\int_K\int_K[\tilde{V}^{2m-1}+(1-\tilde{V})^{2m-1}]\,dP_1\,dP_2\,dP_3=$$

$$=\beta_{2m}\left(1-\frac{2}{2m+1}-\frac{(2m-1)2m}{12}\int_K\int_K\int_K[\tilde{V}^{2m-2}+(1-\tilde{V})^{2m-2}]\,dP_1\,dP_2\,dP_3\right)+$$

$$+\ldots+\beta_{2m-2k+2}\left(1-\frac{2}{2m-2k+3}-\frac{(2m-2k+1)(2m-2k+2)}{12}\cdot\right.$$

$$\left.\cdot\int_K\int_K\int_K[\tilde{V}^{2m-2k}+(1-\tilde{V})^{2m-2k}]\,dP_1\,dP_2\,dP_3\right)+$$

$$+\ldots+\beta_4\left(1-\frac{2}{5}-\frac{3\cdot 4}{12}\int_K\int_K\int_K[\tilde{V}^2+(1-\tilde{V})^2]\,dP_1\,dP_2\,dP_3\right),\tag{3.4}$$

or, by developing the integrants,

$$1-\frac{2}{2m+2}-\frac{2m(2m+1)}{12}\int_K\int_K\int_K\left[\sum_{i=0}^{2m-2}\binom{2m-1}{i}(-1)^i\tilde{V}^i\right]dP_1\,dP_2\,dP_3=$$

$$=\beta_{2m}\left(1-\frac{2}{2m+1}-\frac{2(2m-1)2m}{12}\int_K\int_K\int_K\tilde{V}^{2m-2}\,dP_1\,dP_2\,dP_3-\right.$$

$$\left.-\frac{(2m-1)2m}{12}\int_K\int_K\int_K\left[\sum_{i=0}^{2m-3}\binom{2m-2}{i}(-1)^i\tilde{V}^i\right]dP_1\,dP_2\,dP_3\right)+$$

$$+\ldots+\beta_{2m-2k+2}\left(1-\frac{2}{2m-2k+3}-\frac{2(2m-2k+1)(2m-2k+2)}{12}\cdot\right.$$

$$\cdot\int_K\int_K\int_K\tilde{V}^{2m-2k}\,dP_1\,dP_2\,dP_3-\frac{(2m-2k+1)(2m-2k+2)}{12}\cdot$$

$$\left.\cdot\int_K\int_K\int_K\left[\sum_{i=0}^{2m-2k-1}\binom{2m-2k}{i}(-1)^i\tilde{V}^i\right]dP_1\,dP_2\,dP_3\right)+$$

$$+\ldots+\beta_4\left(1-\frac{2}{5}-2\int_K\int_K\int_K\tilde{V}^2\,dP_1\,dP_2\,dP_3-\int_K\int_K\int_K[-2\tilde{V}+1]\,dP_1\,dP_2\,dP_3\right).\tag{3.5}$$

Comparing the coefficients of $\int_K\int_K\int_K\tilde{V}^i\,dP_1\,dP_2\,dP_3$ $(i=0,1,\ldots,2m-2)$ in (3.5) we get

$$\frac{2m(2m+1)}{12}\binom{2m-1}{2m-2}=\frac{2(2m-1)2m}{12}\beta_{2m},\tag{3.6}$$

$$\frac{2m(2m+1)}{12}\binom{2m-1}{2m-2k}=\frac{(2m-1)2m}{12}\binom{2m-2}{2m-2k}\beta_{2m}+$$

$$+\ldots+\frac{2(2m-2k+1)(2m-2k+2)}{12}\beta_{2m-2k+2}\quad\text{for } k=2,3,\ldots,m-1,\tag{3.7}$$

$$\frac{2m(2m+1)}{12}\binom{2m-1}{2m-2k-1}=\frac{(2m-1)2m}{12}\binom{2m-2}{2m-2k-1}\beta_{2m}+$$
$$+\ldots+\frac{(2m-2k+1)(2m-2k+2)}{12}\binom{2m-2k}{2m-2k-1}\beta_{2m-2k+2}$$
$$\text{for } k=1,2,\ldots,m-1 \tag{3.8}$$

and

$$1-\frac{2}{2m+2}-\frac{2m(2m+1)}{12}=\left(1-\frac{2}{2m+1}-\frac{(2m-1)2m}{12}\right)\beta_{2m}+$$
$$+\ldots+\left(1-\frac{2}{2m-2k+3}-\frac{(2m-2k+1)(2m-2k+2)}{12}\right)\beta_{2m-2k+2}+$$
$$+\ldots+\left(1-\frac{2}{5}-1\right)\beta_4\,. \tag{3.9}$$

From (3.6) and (3.7) we easily conclude that

$$\beta_{2m}=\frac{2m+1}{2} \tag{3.10}$$

and

$$\beta_{2m-2k+2}=\frac{m(2m+1)}{(2m-2k+1)(2m-2k+2)}\cdot$$
$$\cdot\left[\binom{2m-1}{2m-2k}-\sum_{i=1}^{k-1}\frac{(2m-2i+1)(2m-2i+2)}{2m(2m+1)}\binom{2m-2i}{2m-2k}\beta_{2m-2i+2}\right]$$
$$\text{for } k=2,3,\ldots,m-1\,, \tag{3.11}$$

in accordance with (1.5′) and (1.5″).
Finally, expressions (3.8) and (3.9) can be verified by an adequate addition of the constants defined by (3.11).

4. Some remarks

Remark 1. For $m=2$ expressions (1.4) and (1.5) become

$$V_4^{(2)}(K)=2V_3^{(2)}(K) \quad \text{and} \quad V_5^{(3)}(K)=\tfrac{5}{2}V_4^{(3)}(K)\,,$$

coincident with Buchta's formulas (1.2) and (1.3).

Remark 2. Some numerical values for the planar case:

$$V_4^{(2)}(K) = 2\,V_3^{(2)}(K)\,,$$
$$V_6^{(2)}(K) = 3\,V_5^{(2)}(K) - 5\,V_3^{(2)}(K)\,,$$
$$V_8^{(2)}(K) = 4\,V_7^{(2)}(K) - 14\,V_5^{(2)}(K) + 28\,V_3^{(2)}(K)\,,$$
$$V_{10}^{(2)}(K) = 5\,V_9^{(2)}(K) - 30\,V_7^{(2)}(K) + 126\,V_5^{(2)}(K) - 255\,V_3^{(2)}(K)\,.$$

Remark 3. Some numerical values for the three-dimensional case:

$$V_5^{(3)}(K) = \tfrac{5}{2}\,V_4^{(3)}(K)\,,$$
$$V_7^{(3)}(K) = \tfrac{7}{2}\,V_6^{(3)}(K) - \tfrac{35}{4}\,V_4^{(3)}(K)\,,$$
$$V_9^{(3)}(K) = \tfrac{9}{2}\,V_8^{(3)}(K) - 21\,V_6^{(3)}(K) + 63\,V_4^{(3)}(K)\,.$$

Remark 4. Recently, again Buchta [3], p. 96, generalizes his results (1.2) and (1.3) to higher dimensions. For an arbitrary d-dimensional convex body he shows that

$$V_{d+2}^{(d)}(K) = \frac{d+2}{2}\,V_{d+1}^{(d)}(K)\,.$$

Remark 5. Unfortunately, the procedure to relate the expected volume $V_n^{(d)}(K)$, $d = 2, 3$, of the convex hull of n random points to the integral expressions (2.3) and (3.2) cannot be extended to higher dimensions. Therefore it seems to be difficult to generalize our results to higher dimensions.

Fernando Affentranger
Department of Mathematics
University of Buenos Aires

REFERENCES

1 C. Buchta: Ueber die konvexe Huelle von Zufallspunkten in Eibereichen. Elem. Math. *38*, 153–156 (1983).
2 C. Buchta: Zufaellige Polyeder – Eine Uebersicht. Lec. Notes Math. 1114, 1–13. Berlin–Heidelberg–New York–Tokyo. Springer 1985.
3 C. Buchta: On a Conjecture of R. E. Miles About the Convex Hull of Random Points. Mh. Math. *102*, 91–102 (1986).
4 B. Efron: The convex hull of a random set of points. Biometrika *52*, 331–343 (1965).
5 A. Rényi, R. Sulanke: Ueber die konvexe Huelle von n zufaellig gewaehlten Punkten. Z. Wahrscheinlichkeitsth. *2*, 75–84 (1963).
6 L. A. Santaló: Integral Geometry and Geometric Probability. Addison-Wesley. Reading, Mass. (1976).

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