or,

$$\frac{1}{n^2} \frac{V^{\frac{n-2}{n}}(s^{(n)})}{V^{\frac{n-2}{n}}(s'^{(n)})} \geqq \frac{\lambda^{n-1}}{\mu}.$$ (10)

Therefore from (7) and (10) we obtain

$$D^2 \geqq n^4 V^{\frac{2}{n}}(s^{(n)}) V^{\frac{2n-2}{n}}(s'^{(n)}).$$

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A tournament result deduced from harems

There is a large class of difficult problems of the type: "does there exist a graph with n vertices having prescribed degrees $d_1, \ldots, d_n$?" Restricting the problem to particular types of graphs can lead to some very neat characterisations. For example, it is a straightforward exercise to show that a tree exists on $n \, (\geqq 2)$ vertices with degrees $d_1, \ldots, d_n$ if and only if the $d_1, \ldots, d_n$ are positive integers with

$$d_1 + \ldots + d_n = 2(n-1).$$

We shall now restrict attention to 'tournaments'. A *tournament* is a directed graph in which each pair of distinct vertices is joined precisely once (in one direction or the other). Alternatively it can be thought of as a competition of a set of players in which each pair plays once resulting in a win for one of the players. Before proceeding, note that, for example, there exists a tournament of 4 players in which their numbers of wins are 1, 1, 2 and 2 (e.g. A beats B, B beats D, C beats A, C beats B, D beats A and

D beats C). However, there exists no tournament of 6 players with totals of wins 0, 1, 1, 4, 4, 5. One way to see this is to note that any 3 players are involved in a total of $5 + 4 + 3$ games and so certainly the top 3 players cannot win a total of $5 + 4 + 4$. Another way is to see that the bottom 3 players must have won between them at least the 3 matches they played amongst themselves.

Given non-negative integers $w_1, \ldots, w_n$, to be able to construct a tournament of n players with those numbers of wins it is clearly necessary that

$$w_1 + \ldots + w_n = \text{total number of games played} = \tbinom{n}{2} ; \tag{1}$$

any r of the w_i's must add to at most

$$(n - 1) + (n - 2) + \ldots + (n - r) \tag{2}$$

since those r players have only been involved in that number of games;

any s of the w_i's must add to at least $\tbinom{s}{2}$ since those s players will have played $\tbinom{s}{2}$ games amongst themselves. $\tag{3}$

In fact, given condition (1), it is easy to see that (2) and (3) are equivalent. It is a very surprising result that given non-negative integers $w_1, \ldots, w_n$ satisfying (1) and (3) (or (1) and (2)) there *does* exist a tournament of n players with those numbers of wins. This is known as *Landau's theorem* and its proof of the sufficiency of these conditions is usually a fairly involved argument concerning matrices or graphs (see [2] for example). We shall deduce it a much simpler way from a result on marriages.

I shall first remind the reader of Hall's marriage theorem and its generalisation to harems. From this latter result I shall then deduce Landau's theorem proving those necessary and sufficient conditions for a tournament to exist with prescribed numbers of wins for each player.

Hall's theorem, in its marriage form, says that in a set of boys and girls it is possible to find for each boy a different girl whom he knows if and only if any subset B of the boys knows, between them, at least $|B|$ girls. (The female partners can be thought of as wives for the boys.) Various forms of Hall's theorem can be found, for example, in [1] and [3].

The 'harem problem' is a well-known and natural extension of Hall's theorem: it concerns a set of boys requiring various numbers of wives.

Theorem. Let $w_1, \ldots, w_n$ be non-negative integers and assume that boys $b_1, \ldots, b_n$ wish to find, respectively, $w_1, \ldots, w_n$ different female partners whom they know. This can be done if and only if any subset $B = \{b_{i_1}, \ldots, b_{i_r}\}$ of the boys knows, between them, at least $w_{i_1} + \ldots + w_{i_r}$ girls.

Proof. Consider a new set of boys in which each of the original boys b_i is duplicated to give w_i copies of that boy, and such that each duplicated boy has the same girl-friends as the original. Then it is not hard to see that

in the original situation the boys $b_1, \ldots, b_n$ can be found $w_1, \ldots, w_n$ 'wives' respectively

$\Leftrightarrow$ in the new situation each boy can be found a wife

$\Leftrightarrow$ in the new situation any set B^* of boys knows at least $|B^*|$ girls (Hall's theorem)

$\Leftrightarrow$ in the original situation any set $B = \{b_{i_1}, \ldots, b_{i_r}\}$ of boys knows at least $w_{i_1} + \ldots + w_{i_r}$ girls.

Hence the harem result follows.

We are now able to deduce our results on tournaments.

Theorem. Let $w_1, \ldots, w_n$ be non-negative integers whose sum is $\binom{n}{2}$. Then a tournament of n players exists in which the players win totals of $w_1, \ldots, w_n$ games, respectively, if and only if any r of the w_i's add to at most

$$(n-1) + (n-2) + \ldots + (n-r).$$

Proof. Consider a collection of n boys $b_1, \ldots, b_n$ and $\binom{n}{2}$ girls $g_{1,1}, g_{1,2}, \ldots, g_{n-1,n}$ such that girl $g_{i,j}$ is known by just boys b_i and b_j. By regarding boy b_i (or b_j) marrying girl $g_{i,j}$ as the ith (or jth) player winning the game between players i and j it is clear that we can switch from tournaments to marriage (and back) and deduce that

there exists a tournament of n players winning $w_1, \ldots, w_n$ games

$\Leftrightarrow$ in the new boy/girl situation the boys $b_1, \ldots, b_n$ can be found $w_1, \ldots, w_n$ wives, respectively

$\Leftrightarrow$ in the new boy/girl situation any set of boys $\{b_{i_1}, \ldots, b_{i_r}\}$ knows between them at least $w_{i_1} + \ldots + w_{i_r}$ girls (the harem result)

[Note that, by the construction of the boy/girl relationships, r boys know between them exactly

$$(n-1) + (n-2) + \ldots + (n-r)$$

girls.]

$\Leftrightarrow$ in the new boy/girl situation any set of boys $\{b_{i_1}, \ldots, b_{i_r}\}$ satisfies

$$(n-1) + \ldots + (n-r) \geq w_{i_1} + \ldots + w_{i_r}$$

$\Leftrightarrow$ any r of the w_i's add to at most

$$(n-1) + (n-2) + \ldots + (n-r).$$

The required result follows.

Corollary (Landau's theorem). Let $w_1, \ldots, w_n$ be non-negative integers whose sum is $\binom{n}{2}$. Then a tournament of n players exists in which the players win totals of $w_1, \ldots, w_n$ games, respectively, if and only if any s of the w_i's add to at least $\binom{s}{2}$.

Proof. Any s of the w_i's add to at least $\binom{s}{2}$

$\Leftrightarrow$ any $n - s$ of the w_i's add to at most $\binom{n}{2} - \binom{s}{2}$

$\Leftrightarrow$ any r of the w_i's add to at most $\binom{n}{2} - \binom{n-r}{2}$

$= [1 + 2 + \ldots + (n-1)] - [1 + 2 + \ldots + (n-r-1)]$

$= (n-1) + (n-2) + \ldots + (n-r)$.

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Didaktik und Elementarmathematik

Kombinatorik mit dem Computer: Partitionen und Frankaturen

1. Einleitung

In den verschiedensten Gebieten der Mathematik sieht man sich vor Probleme
gestellt, die auf die Abzählung oder die Auflistung bestimmter Partitionen hinaus-
laufen. Als Partitionen bezeichnet man in der Kombinatorik additive Zerfällungen
einer natürlichen Zahl n mit Summanden aus einer vorgegebenen Referenz-Menge,
wobei die Reihenfolge der Summanden in einer solchen Figur belanglos ist. Die
folgende Aufzählung zeigt die möglichen Partitionen der Zahl 20 über der Referenz-
Menge $\{2, 3, 7\}$:

```
3+3+7+7
2+3+3+3+3+3+3
2+2+3+3+3+7
2+2+2+7+7
2+2+2+2+3+3+3+3

2+2+2+2+2+3+7
2+2+2+2+2+2+2+3+3
2+2+2+2+2+2+2+2+2+2

ANZAHL DER FIGUREN: 8
```

Die Summanden in den einzelnen Figuren sind hier nach zunehmenden Werten
geordnet.
Kommt in einer Partition der Zahl n über der Referenz-Menge

$$\{a_1, a_2, \ldots, a_p\}$$