Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 42 (1987)
Heft: 6

Artikel: A tournament result deduced from harems
Autor: Bryant, V.W.
DOI: https://doi.org/10.5169/seals-40044

or,

$$\frac{1}{n^2}\frac{V^{\frac{n-2}{n}}(s^{(n)})}{V^{\frac{n-2}{n}}(s'^{(n)})} \geqq \frac{\lambda^{n-1}}{\mu}. \tag{10}$$

Therefore from (7) and (10) we obtain

$$D^2 \geqq n^4 V^{\frac{2}{n}}(s^{(n)})\, V^{\frac{2n-2}{n}}(s'^{(n)}).$$

The author is grateful to the referee for his helpful suggestions.

G. Tsintsifas, Thessaloniki, Greece

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 0013-6018/87/060000-00$1.50+0.20/0

A tournament result deduced from harems

There is a large class of difficult problems of the type: "does there exist a graph with n vertices having prescribed degrees $d_1, \ldots, d_n$?" Restricting the problem to particular types of graphs can lead to some very neat characterisations. For example, it is a straightforward exercise to show that a tree exists on $n\,(\geqq 2)$ vertices with degrees $d_1, \ldots, d_n$ if and only if the $d_1, \ldots, d_n$ are positive integers with

$$d_1 + \ldots + d_n = 2(n-1).$$

We shall now restrict attention to 'tournaments'. A *tournament* is a directed graph in which each pair of distinct vertices is joined precisely once (in one direction or the other). Alternatively it can be thought of as a competition of a set of players in which each pair plays once resulting in a win for one of the players. Before proceeding, note that, for example, there exists a tournament of 4 players in which their numbers of wins are 1, 1, 2 and 2 (e.g. A beats B, B beats D, C beats A, C beats B, D beats A and

D beats C). However, there exists no tournament of 6 players with totals of wins 0, 1, 1, 4, 4, 5. One way to see this is to note that any 3 players are involved in a total of $5+4+3$ games and so certainly the top 3 players cannot win a total of $5+4+4$. Another way is to see that the bottom 3 players must have won between them at least the 3 matches they played amongst themselves.

Given non-negative integers $w_1, \ldots, w_n$, to be able to construct a tournament of n players with those numbers of wins it is clearly necessary that

$$w_1 + \ldots + w_n = \text{total number of games played} = \tbinom{n}{2}\,; \tag{1}$$

any r of the w_i's must add to at most

$$(n-1)+(n-2)+\ldots+(n-r) \tag{2}$$

since those r players have only been involved in that number of games;

any s of the w_i's must add to at least $\binom{s}{2}$ since those s players will have played $\binom{s}{2}$ games amongst themselves. (3)

In fact, given condition (1), it is easy to see that (2) and (3) are equivalent. It is a very surprising result that given non-negative integers $w_1,, \ldots, w_n$ satisfying (1) and (3) (or (1) and (2)) there *does* exist a tournament of n players with those numbers of wins. This is known as *Landau's theorem* and its proof of the sufficiency of these conditions is usually a fairly involved argument concerning matrices or graphs (see [2] for example). We shall deduce it a much simpler way from a result on marriages.

I shall first remind the reader of Hall's marriage theorem and its generalisation to harems. From this latter result I shall then deduce Landau's theorem proving those necessary and sufficient conditions for a tournament to exist with prescribed numbers of wins for each player.

Hall's theorem, in its marriage form, says that in a set of boys and girls it is possible to find for each boy a different girl whom he knows if and only if any subset B of the boys knows, between them, at least $|B|$ girls. (The female partners can be thought of as wives for the boys.) Various forms of Hall's theorem can be found, for example, in [1] and [3].

The 'harem problem' is a well-known and natural extension of Hall's theorem: it concerns a set of boys requiring various numbers of wives.

Theorem. Let $w_1, \ldots, w_n$ be non-negative integers and assume that boys $b_1, \ldots, b_n$ wish to find, respectively, $w_1, \ldots, w_n$ different female partners whom they know. This can be done if and only if any subset $B = \{b_{i_1}, \ldots, b_{i_r}\}$ of the boys knows, between them, at least $w_{i_1} + \ldots + w_{i_r}$ girls.

Proof. Consider a new set of boys in which each of the original boys b_i is duplicated to give w_i copies of that boy, and such that each duplicated boy has the same girl-friends as the original. Then it is not hard to see that in the original situation the boys $b_1, \ldots, b_n$ can be found $w_1, \ldots, w_n$ 'wives' respectively

$\Leftrightarrow$ in the new situation each boy can be found a wife

$\Leftrightarrow$ in the new situation any set B^* of boys knows at least $|B^*|$ girls (Hall's theorem)

$\Leftrightarrow$ in the original situation any set $B = \{b_{i_1}, \ldots, b_{i_r}\}$ of boys knows at least $w_{i_1} + \ldots + w_{i_r}$ girls.

Hence the harem result follows.

We are now able to deduce our results on tournaments.

Theorem. Let $w_1, \ldots, w_n$ be non-negative integers whose sum is $\binom{n}{2}$. Then a tournament of n players exists in which the players win totals of $w_1, \ldots, w_n$ games, respectively, if and only if any r of the w_i's add to at most

$$(n-1) + (n-2) + \ldots + (n-r).$$

Proof. Consider a collection of n boys $b_1, \ldots, b_n$ and $\binom{n}{2}$ girls $g_{1,1}, g_{1,2}, \ldots, g_{n-1,n}$ such that girl $g_{i,j}$ is known by just boys b_i and b_j. By regarding boy b_i (or b_j) marrying girl $g_{i,j}$ as the ith (or jth) player winning the game between players i and j it is clear that we can switch from tournaments to marriage (and back) and deduce that

there exists a tournament of n players winning $w_1, \ldots, w_n$ games

$\Leftrightarrow$ in the new boy/girl situation the boys $b_1, \ldots, b_n$ can be found $w_1, \ldots, w_n$ wives, respectively

$\Leftrightarrow$ in the new boy/girl situation any set of boys $\{b_{i_1}, \ldots, b_{i_r}\}$ knows between them at least $w_{i_1} + \ldots + w_{i_r}$ girls (the harem result)
[Note that, by the construction of the boy/girl relationships, r boys know between them exactly

$$(n-1) + (n-2) + \ldots + (n-r)$$

girls.]

$\Leftrightarrow$ in the new boy/girl situation any set of boys $\{b_{i_1}, \ldots, b_{i_r}\}$ satisfies

$$(n-1) + \ldots + (n-r) \geqq w_{i_1} + \ldots + w_{i_r}$$

$\Leftrightarrow$ any r of the w_i's add to at most

$$(n-1) + (n-2) + \ldots + (n-r).$$

The required result follows.

Corollary (Landau's theorem). Let $w_1, \ldots, w_n$ be non-negative integers whose sum is $\binom{n}{2}$. Then a tournament of n players exists in which the players win totals of $w_1, \ldots, w_n$ games, respectively, if and only if any s of the w_i's add to at least $\binom{s}{2}$.

Proof. Any s of the w_i's add to at least $\binom{s}{2}$
$\Leftrightarrow$ any $n-s$ of the w_i's add to at most $\binom{n}{2} - \binom{s}{2}$
$\Leftrightarrow$ any r of the w_i's add to at most $\binom{n}{2} - \binom{n-r}{2}$
$= [1 + 2 + \ldots + (n-1)] - [1 + 2 + \ldots + (n-r-1)]$
$= (n-1) + (n-2) + \ldots + (n-r)$.

V. W. Bryant, Department of Pure Mathematics,
Sheffield University

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 0013-6018/87/060000-00$1.50+0.20/0

Didaktik und Elementarmathematik

Kombinatorik mit dem Computer: Partitionen und Frankaturen

1. Einleitung

In den verschiedensten Gebieten der Mathematik sieht man sich vor Probleme gestellt, die auf die Abzählung oder die Auflistung bestimmter Partitionen hinauslaufen. Als Partitionen bezeichnet man in der Kombinatorik additive Zerfällungen einer natürlichen Zahl n mit Summanden aus einer vorgegebenen Referenz-Menge, wobei die Reihenfolge der Summanden in einer solchen Figur belanglos ist. Die folgende Aufzählung zeigt die möglichen Partitionen der Zahl 20 über der Referenz-Menge $\{2, 3, 7\}$:

```
3+3+7+7
2+3+3+3+3+3+3
2+2+3+3+3+7
2+2+2+7+7
2+2+2+2+3+3+3+3

2+2+2+2+2+3+7
2+2+2+2+2+2+2+3+3
2+2+2+2+2+2+2+2+2+2

ANZAHL DER FIGUREN: 8
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Die Summanden in den einzelnen Figuren sind hier nach zunehmenden Werten geordnet.
Kommt in einer Partition der Zahl n über der Referenz-Menge

$$\{a_1, a_2, \ldots, a_p\}$$