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## A generalization of a two triangle inequality

### 1. Introduction

By coupling an inequality of Bottema [1] together with one of Pedoe [1], O. Bottema and M. S. Klamkin obtained the two chain inequality

$$a'x + b'y + c'z \cong \left[ \frac{P}{2} + 8FF' \right]^{1/2} \cong 4\sqrt{FF'},$$

see [2], where  $P = \sum a'^2(b^2 + c^2 - a^2)$ ,  $x, y, z$  the distances of an interior point  $M$  of the triangle  $ABC$  from the vertices  $A, B, C$ ,  $a, b, c$  and  $a', b', c'$  the sides of the triangles  $ABC$  and  $A'B'C'$  and  $F, F'$  their area respectively.

In this note the author will generalize the part

$$a'x + b'y + c'z \cong 4\sqrt{FF'}$$

of the above inequality for two simplices  $s^{(n)} = (A_1 A_2 \dots A_{n+1})$  and  $s'^{(n)} = (A'_1 A'_2 \dots A'_{n+1})$ .

### 2. Notations

We denote by  $V(W)$  the volume of the simplex  $W$ ,  $s^{(n)} = (A_1 A_2 \dots A_{n+1})$ ,  $s'^{(n)} = (A'_1 A'_2 \dots A'_{n+1})$  two simplices of  $E^n$ . The facets

$$(A_1 A_2 \dots A_{i-1} A_{i+1} \dots A_{n+1}), (A'_1 A'_2 \dots A'_{i-1} A'_{i+1} \dots A'_{n+1})$$

will be denoted by  $s_i^{(n-1)}$ ,  $s'_i{}^{(n-1)}$  respectively.

Suppose that  $M$  is an interior point of the simplex  $s^{(n)}$  with distances  $A_i M = x_i$ . We put:

$$D = \sum_{i=1}^{n+1} x_i V(s'_i{}^{(n-1)}).$$

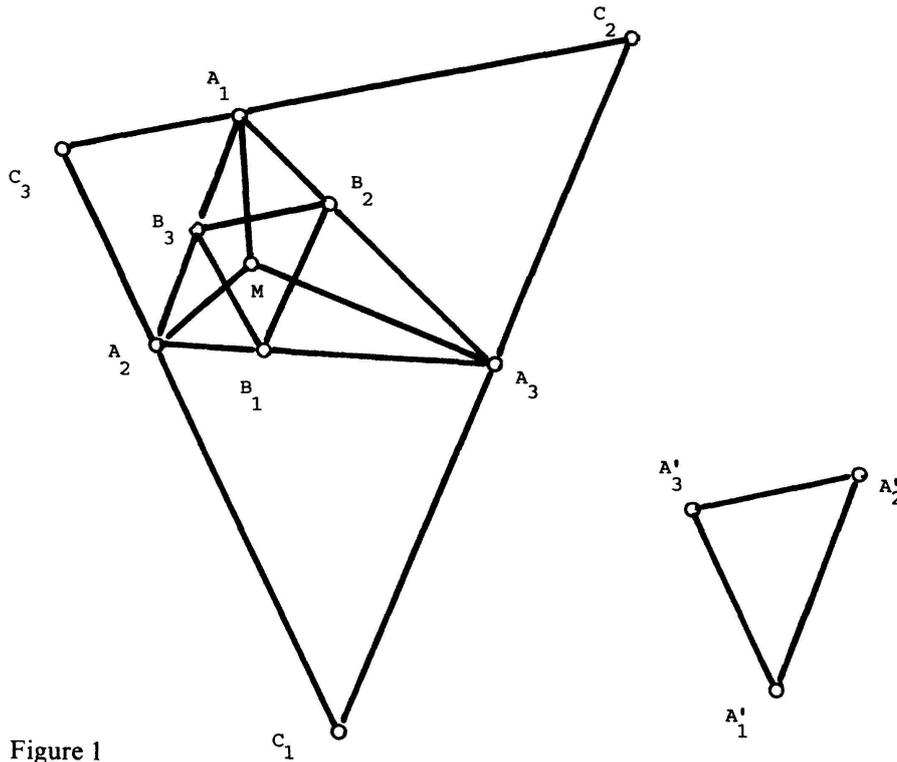


Figure 1

The quantity  $D$  depends on the labeling of the vertex set but for every case we can prove:

### 3. Theorem

For two simplices  $s^{(n)}$  and  $s'^{(n)}$  holds:

$$D^2 \geq n^4 \cdot V^{\frac{n}{2}}(s^{(n)}) \cdot V^{\frac{2n-2}{2}}(s'^{(n)}) .$$

*Proof:* We assume that the simplex  $p^{(n)} = (B_1 B_2 \dots B_n)$  is similar to  $s'^{(n)}$  and inscribed in  $s^{(n)}$ , so that  $B_i \in s_i^{(n-1)}$ . We also consider the simplex  $q^{(n)} = (C_1 C_2 \dots C_{n+1})$  similar to  $s'^{(n)}$  and circumscribed to  $s^{(n)}$  so that  $A_i \in q_i^{(n-1)}$ , where  $q_i^{(n-1)} = (C_1 C_2 \dots C_{i-1} C_{i+1} \dots C_{n+1})$ . It is known that:

$$V^n(s^{(n)}) \geq V^{n-1}(p^{(n)}) \cdot V(q^{(n)}) , \tag{1}$$

see [3].

Suppose  $W' = kW$  is the homothetical image of  $W$  with ratio  $k$ . We put:

$$p^{(n)} = \lambda s'^{(n)} \quad \text{and} \quad q^{(n)} = \mu s'^{(n)} . \tag{2}$$

We can easily see that:  $x_i \cdot V(p_i^{(n-1)}) \cong nV(A_i B_1 B_2 \dots B_{i-1} B_{i+1} \dots B_{n+1} M)$  and  $x_i \cdot V(q_i^{(n-1)}) \cong nV(C_1 C_2 \dots C_{i-1} C_{i+1} \dots C_{n+1} M)$ . From the above follows

$$\sum_{i=1}^{n+1} x_i V(p_i^{(n-1)}) \cong nV(s^{(n)}), \tag{3}$$

$$\sum_{i=1}^{n+1} x_i V(q_i^{(n-1)}) \cong nV(q^n), \tag{4}$$

where  $p_i^{(n-1)} = (B_1 B_2 \dots B_{i-1} B_{i+1} \dots B_{n+1})$ .

See the figure, above, for the elementary case  $n = 2$ . From (2) follows,

$$V(p_i^{(n-1)}) = \lambda^{n-1} V(s'^{(n-1)}), \tag{5a}$$

$$V(q_i^{(n-1)}) = \mu^{n-1} V(s'^{(n-1)}) \quad \text{and} \tag{5b}$$

$$V(q^{(n)}) = \mu^n V(s'^{(n)}), \tag{5c}$$

see (2).

Therefore from (3), (4), (5) we have:

$$\lambda^{n-1} D \cong nV(s^{(n)}), \quad \mu^{n-1} D \cong n\mu^n V(s'^{(n)}) \tag{6}$$

or,

$$\frac{\lambda^{n-1}}{\mu} D^2 \cong n^2 V(s^{(n)}) V(s'^{(n)}). \tag{7}$$

Using (1), (5c) we take:

$$\frac{V^n(s^{(n)})}{V^2(q^{(n)})} \cong \frac{\lambda^{n(n-1)} V^{n-1}(s'^{(n)})}{\mu^n V(s'^{(n)})} = \left(\frac{\lambda^{n-1}}{\mu}\right)^n V^{n-2}(s'^{(n)})$$

or,

$$V^{n-2}(s^{(n)}) \left(\frac{V(s^{(n)})}{V(q^{(n)})}\right)^2 \cong \left(\frac{\lambda^{n-1}}{\mu}\right)^n V^{n-2}(s'^{(n)}). \tag{8}$$

But it is known that if  $V(q^{(n)})$  is minimum then  $A_i$  is the centroid of  $q_i^{(n-1)}$ , see [4] or [5], therefore

$$\frac{1}{n^n} \cong \frac{V(s^{(n)})}{V(q^{(n)})}. \tag{9}$$

From (8), (9) follows:

$$\frac{V^{n-2}(s^{(n)})}{n^{2n}} \cong \left(\frac{\lambda^{n-1}}{\mu}\right)^n V^{n-2}(s'^{(n)})$$

or,

$$\frac{1}{n^2} \frac{V^{\frac{n-2}{n}}(s^{(n)})}{V^{\frac{n-2}{n}}(s'(n))} \geq \frac{\lambda^{n-1}}{\mu}. \tag{10}$$

Therefore from (7) and (10) we obtain

$$D^2 \geq n^4 V^{\frac{2}{n}}(s^{(n)}) V^{\frac{2n-2}{n}}(s'(n)).$$

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## A tournament result deduced from harems

There is a large class of difficult problems of the type: “does there exist a graph with  $n$  vertices having prescribed degrees  $d_1, \dots, d_n$ ?” Restricting the problem to particular types of graphs can lead to some very neat characterisations. For example, it is a straightforward exercise to show that a tree exists on  $n (\geq 2)$  vertices with degrees  $d_1, \dots, d_n$  if and only if the  $d_1, \dots, d_n$  are positive integers with

$$d_1 + \dots + d_n = 2(n - 1).$$

We shall now restrict attention to ‘tournaments’. A *tournament* is a directed graph in which each pair of distinct vertices is joined precisely once (in one direction or the other). Alternatively it can be thought of as a competition of a set of players in which each pair plays once resulting in a win for one of the players. Before proceeding, note that, for example, there exists a tournament of 4 players in which their numbers of wins are 1, 1, 2 and 2 (e.g. A beats B, B beats D, C beats A, C beats B, D beats A and