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A generalization of a two triangle inequality

1. Introduction

By coupling an inequality of Bottema [1] together with one of Pedoe [1], O. Bottema and M. S. Klamkin obtained the two chain inequality

$$a'x + b'y + c'z \geqq \left[\frac{P}{2} + 8FF'\right]^{1/2} \geqq 4\sqrt{FF'}\,,$$

see [2], where $P = \sum a'^2(b^2 + c^2 - a^2)$, x, y, z the distances of an interior point M of the triangle ABC from the vertices A, B, C, a, b, c and a', b', c' the sides of the triangles ABC and $A'B'C'$ and F, F' their area respectively.
In this note the author will generalize the part

$$a'x + b'y + c'z \geqq 4\sqrt{FF'}$$

of the above inequality for two simplices $s^{(n)} = (A_1A_2\ldots A_{n+1})$ and $s'^{(n)} = (A'_1A'_2\ldots A'_{n+1})$.

2. Notations

We denote by $V(W)$ the volume of the simplex W, $s^{(n)} = (A_1A_2\ldots A_{n+1})$, $s'^{(n)} = (A'_1A'_2\ldots A'_{n+1})$ two simplices of E^n. The facets

$$(A_1A_2\ldots A_{i-1}A_{i+1}\ldots A_{n+1}),\ (A'_1A'_2\ldots A'_{i-1}A'_{i+1}\ldots A'_{n+1})$$

will be denoted by $s_i^{(n-1)}$, $s_i'^{(n-1)}$ respectively.
Suppose that M is an interior point of the simplex $s^{(n)}$ with distances $A_iM = x_i$. We put:

$$D = \sum_{i=1}^{n+1} x_i V(s_i'^{(n-1)})\,.$$

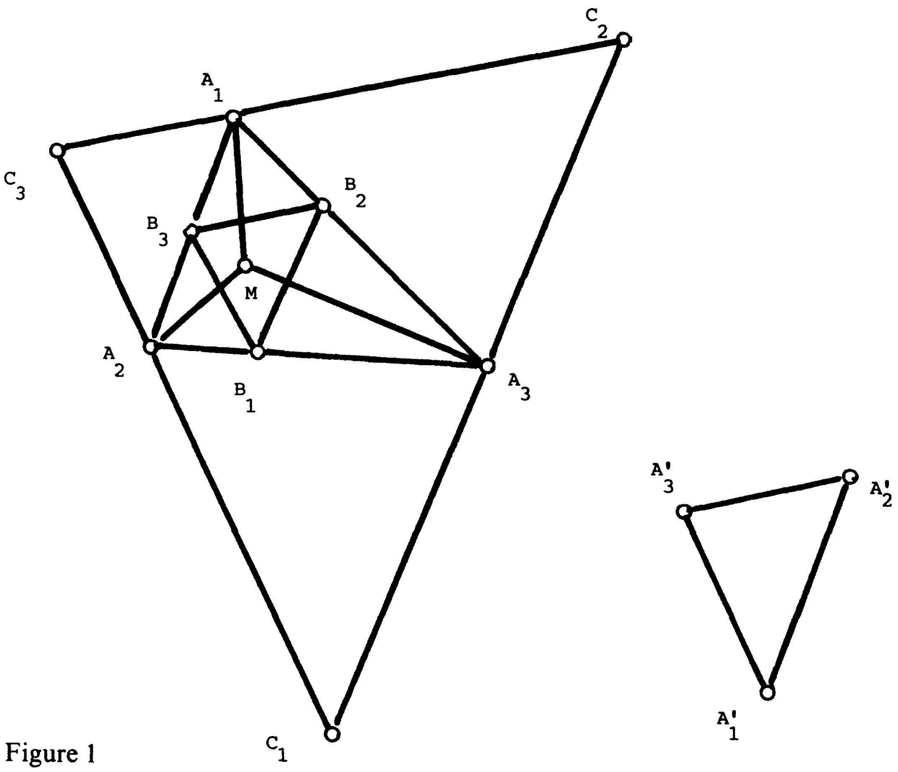

Figure 1

The quantity D depends on the labeling of the vertex set but for every case we can prove:

3. Theorem

For two simplices $s^{(n)}$ and $s'^{(n)}$ holds:

$$D^2 \geqq n^4 \cdot V^{\frac{n}{2}}(s^{(n)}) \cdot V^{\frac{2n-2}{2}}(s'^{(n)}) .$$

Proof: We assume that the simplex $p^{(n)} = (B_1 B_2 \ldots B_n)$ is similar to $s'^{(n)}$ and inscribed in $s^{(n)}$, so that $B_i \in s_i^{(n-1)}$. We also consider the simplex $q^{(n)} = (C_1 C_2 \ldots C_{n+1})$ similar to $s'^{(n)}$ and circumscribed to $s^{(n)}$ so that $A_i \in q_i^{(n-1)}$, where $q_i^{(n-1)} = (C_1 C_2 \ldots C_{i-1} C_{i+1} \ldots C_{n+1})$. It is known that:

$$V^n(s^{(n)}) \geqq V^{n-1}(p^{(n)}) \cdot V(q^{(n)}) , \qquad (1)$$

see [3].
Suppose $W' = kW$ is the homothetical image of W with ratio k. We put:

$$p^{(n)} = \lambda s'^{(n)} \quad \text{and} \quad q^{(n)} = \mu s'^{(n)} . \qquad (2)$$

We can easily see that: $x_i \cdot V(p_i^{(n-1)}) \geqq n V(A_i B_1 B_2 \ldots B_{i-1} B_{i+1} \ldots B_{n+1} M)$ and $x_i \cdot V(q_i^{(n-1)}) \geqq n V(C_1 C_2 \ldots C_{i-1} C_{i+1} \ldots C_{n+1} M)$. From the above follows

$$\sum_{i=1}^{n+1} x_i V(p_i^{(n-1)}) \geqq n V(s^{(n)}), \tag{3}$$

$$\sum_{i=1}^{n+1} x_i V(q_i^{(n-1)}) \geqq n V(q^n), \tag{4}$$

where $p_i^{(n-1)} = (B_1 B_2 \ldots B_{i-1} B_{i+1} \ldots B_{n+1})$.
See the figure, above, for the elementary case $n = 2$. From (2) follows,

$$V(p_i^{(n-1)}) = \lambda^{n-1} V(s'^{(n-1)}), \tag{5a}$$

$$V(q_i^{(n-1)}) = \mu^{n-1} V(s'^{(n-1)}) \quad \text{and} \tag{5b}$$

$$V(q^{(n)}) = \mu^n V(s'^{(n)}), \tag{5c}$$

see (2).
Therefore from (3), (4), (5) we have:

$$\lambda^{n-1} D \geqq n V(s^{(n)}), \quad \mu^{n-1} D \geqq n \mu^n V(s'^{(n)}) \tag{6}$$

or,

$$\frac{\lambda^{n-1}}{\mu} D^2 \geqq n^2 V(s^{(n)}) V(s'^{(n)}). \tag{7}$$

Using (1), (5c) we take:

$$\frac{V^n(s^{(n)})}{V^2(q^{(n)})} \geqq \frac{\lambda^{n(n-1)} V^{n-1}(s'^{(n)})}{\mu^n V(s'^{(n)})} = \left(\frac{\lambda^{n-1}}{\mu}\right)^n V^{n-2}(s'^{(n)})$$

or,

$$V^{n-2}(s^{(n)}) \left(\frac{V(s^{(n)})}{V(q^{(n)})}\right)^2 \geqq \left(\frac{\lambda^{n-1}}{\mu}\right)^n V^{n-2}(s'^{(n)}). \tag{8}$$

But it is known that if $V(q^{(n)})$ is minimum then A_i is the centroid of $q_i^{(n-1)}$, see [4] or [5], therefore

$$\frac{1}{n^n} \geqq \frac{V(s^{(n)})}{V(q^{(n)})}. \tag{9}$$

From (8), (9) follows:

$$\frac{V^{n-2}(s^{(n)})}{n^{2n}} \geqq \left(\frac{\lambda^{n-1}}{\mu}\right)^n V^{n-2}(s'^{(n)})$$

or,

$$\frac{1}{n^2}\frac{V^{\frac{n-2}{n}}(s^{(n)})}{V^{\frac{n-2}{n}}(s'^{(n)})} \geqq \frac{\lambda^{n-1}}{\mu}. \tag{10}$$

Therefore from (7) and (10) we obtain

$$D^2 \geqq n^4 V^{\frac{2}{n}}(s^{(n)})\, V^{\frac{2n-2}{n}}(s'^{(n)}).$$

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A tournament result deduced from harems

There is a large class of difficult problems of the type: "does there exist a graph with n vertices having prescribed degrees $d_1, \ldots, d_n$?" Restricting the problem to particular types of graphs can lead to some very neat characterisations. For example, it is a straightforward exercise to show that a tree exists on $n\,(\geqq 2)$ vertices with degrees $d_1, \ldots, d_n$ if and only if the $d_1, \ldots, d_n$ are positive integers with

$$d_1 + \ldots + d_n = 2(n-1).$$

We shall now restrict attention to 'tournaments'. A *tournament* is a directed graph in which each pair of distinct vertices is joined precisely once (in one direction or the other). Alternatively it can be thought of as a competition of a set of players in which each pair plays once resulting in a win for one of the players. Before proceeding, note that, for example, there exists a tournament of 4 players in which their numbers of wins are 1, 1, 2 and 2 (e.g. A beats B, B beats D, C beats A, C beats B, D beats A and