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Kleine Mitteilung

Note on the diophantine equation $1 + x + x^2 + \ldots + x^n = y^m$

This equation has occurred from time to time in the literature. In this note, we shall treat some aspects of the lowest cases $m = 2$ and $m = 3$. Throughout we shall assume $|x| > 1$ and $n > 1$.

Theorem. *The Diophantine equation* $1 + x + \ldots + x^n = y^2$ *has solutions only for* $n = 3$ *and* $n = 4$, *the solutions being* $(7, \pm 20)$ *and* $(3, \pm 11)$, *respectively. The Diophantine equation* $1 + x + \ldots + x^n = y^3$ *has solutions only for* $n = 2$ *unless n is of the form* $6k + 4$. *The respective solutions are* $(18, 7)$ *and* $(-19, 7)$.

This was shown by W. Ljunggren [4] in 1943*, using results by D. Schepel [7] and T. Nagell [6]. Actually, the case $n = 3$ for the first equation follows from a result already known to Fermat and subsequently proved by E. Lucas and A. Genocchi [2]. (See also [1], vol. II, p. 487.)

Considering only primes for x, the sum $1 + x + \ldots + x^n$ can be interpreted as $\sigma(p^n)$, the sum of the divisors of p^n. The question for $m = 2$ now reads: can $\sigma(p^n)$ ever be a square? It is easily seen that for $n = 1$, $1 + p = y^2$ if and only if $p = 3$, $y = 2$. Hence the answer to our question is simply, that $\sigma(p^n)$ is a square only in the cases $p = 3$, $n = 1$ or 4, and $p = 7$, $n = 3$.

Recently, Takaku [8] has shown that $\sigma(p^n)$ a square requires p to be less than $2^{2^{n+1}}$. In view of Ljunggren's theorem this is quite staggering.

Returning to the second part of the theorem, let us remark that R. Guy ([3], p. 7) poses the following question on so-called repunits, i.e. numbers $11111\ldots11 = 1 + 10 + \ldots + 10^n$: Can such a number ever be a cube? (It is known that, except 1, they can never be squares.) As $1 + 10$ is not a cube, a positive answer to Guy's question can only stem from $n = 6k + 4$. Using the elementary geometric summation formula, one obtains $10^{6k+6} - 10 = 90y^3$, which gives $y^3 \equiv 2 \bmod 7$ and hence is impossible. Therefore, the answer to Guy's question is «no».

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