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## Kleine Mitteilung

### Note on the diophantine equation $1 + x + x^2 + \dots + x^n = y^m$

This equation has occurred from time to time in the literature. In this note, we shall treat some aspects of the lowest cases  $m=2$  and  $m=3$ . Throughout we shall assume  $|x| > 1$  and  $n > 1$ .

**Theorem.** *The Diophantine equation  $1 + x + \dots + x^n = y^2$  has solutions only for  $n=3$  and  $n=4$ , the solutions being  $(7, \pm 20)$  and  $(3, \pm 11)$ , respectively. The Diophantine equation  $1 + x + \dots + x^n = y^3$  has solutions only for  $n=2$  unless  $n$  is of the form  $6k+4$ . The respective solutions are  $(18, 7)$  and  $(-19, 7)$ .*

This was shown by W. Ljunggren [4] in 1943\*, using results by D. Schepel [7] and T. Nagell [6]. Actually, the case  $n=3$  for the first equation follows from a result already known to Fermat and subsequently proved by E. Lucas and A. Genocchi [2]. (See also [1], vol. II, p. 487.)

Considering only primes for  $x$ , the sum  $1 + x + \dots + x^n$  can be interpreted as  $\sigma(p^n)$ , the sum of the divisors of  $p^n$ . The question for  $m=2$  now reads: can  $\sigma(p^n)$  ever be a square? It is easily seen that for  $n=1$ ,  $1 + p = y^2$  if and only if  $p=3$ ,  $y=2$ . Hence the answer to our question is simply, that  $\sigma(p^n)$  is a square only in the cases  $p=3$ ,  $n=1$  or 4, and  $p=7$ ,  $n=3$ .

Recently, Takaku [8] has shown that  $\sigma(p^n)$  a square requires  $p$  to be less than  $2^{2^{n+1}}$ . In view of Ljunggren's theorem this is quite staggering.

Returning to the second part of the theorem, let us remark that R. Guy ([3], p. 7) poses the following question on so-called repunits, i.e. numbers  $1111\dots 11 = 1 + 10 + \dots + 10^n$ : Can such a number ever be a cube? (It is known that, except 1, they can never be squares.) As  $1 + 10$  is not a cube, a positive answer to Guy's question can only stem from  $n = 6k + 4$ . Using the elementary geometric summation formula, one obtains  $10^{6k+6} - 10 = 90y^3$ , which gives  $y^3 \equiv 2 \pmod{7}$  and hence is impossible. Therefore, the answer to Guy's question is «no».

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1\* The author wishes to thank Prof. J. Brzezinski for a copy of Ljunggren's paper which was not available in Poland.