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Some characterizations of complex normed Q-algebras

A complex normed algebra A with unit 1 is a normed vector space over \mathbb{C} where a multiplication $A \times A \to A$ is defined such that

$$||x \cdot y|| \le ||x|| ||y||$$
, for all $x, y \in A$,

where \cdot denotes the multiplication and $\| \|$ the norm on A. With respect to this multiplication the unit 1 of A has of course the property $1 \cdot x = x \cdot 1 = x$ for every $x \in A$.

The normed algebras we usually meet in the applications and in introductory functional analysis courses are the so-called *Banach algebras*. These are normed algebras such that the norm induces a *complete* topology (i.e. one such that Cauchy sequences converge).

Now, every second year mathematics student knows that the set of all invertible elements of a Banach algebra A is open $(x \in A \text{ is invertible})$ if there is a $y \in A$ such that $x \cdot y = y \cdot x = 1$. In such case we write $y =: x^{-1}$. The set of invertibles of A is denoted by Inv (A)). The point is that the converse of this statement is false: just take a look at the algebra R(D) of all complex rational functions defined on the closed unit disk of \mathbb{C} , endowed with the norm $||q|| := \sup_{z \in \mathbb{C}} |q(z)|$. $q \in R(D)$ is invertible if and only if it has

no zeros in D, and thus Inv (R(D)) is open in R(D), by the Maximum Principle. On the other hand, R(D) is clearly no Banach algebra since there are analytic functions on D which are not rational (e.g. $\sin(z)$!).

Since the condition that Inv(A) be open has a well-mixed topological and algebraic nature, it seems interesting to define *Q-algebras* (or *open algebras*, as they are sometimes called) as those algebras (with unit!) which satisfy it. Of course, all Banach algebras are Q-algebras.

Our purpose is to show that almost all fundamental properties of Banach algebras are shared by the larger class of normed Q-algebras. Quite surprisingly, it turns out that some of these properties do actually characterize the normed Q-algebras among the normed algebras with unit.

In the following A will be a complex normed algebra with unit 1 and norm $\|\cdot\|$. Fuster and Marquina [3] have proved the equivalence of the statements

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(Q) A is a Q-algebra
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$$(Q_{\mathsf{FMI}}) \ \exists \ \delta \in (0,1] : x \in A \ \text{and} \ \|1-x\| < \delta \ \text{imply} \ x \in \mathsf{Inv}(A)$$

$$(Q_{\text{FM2}}) \exists \delta \in (0, 1] : x \in A \text{ and } ||x|| < \delta \text{ imply that } \sum x^n \text{ converges in } A.$$

In an unpublished paper [4] Th. W. Palmer has given a further characterization:

 (Q_P) A is inverse-closed in its completion, that is, if A^* is the completion of A, then $Inv(A^*) \cap A \subset Inv(A)$.

Let's now state our theorem:

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Theorem. Let A be a complex normed algebra with unit 1. Then the following conditions are equivalent:

- (Q) A is a Q-algebra
- (Q_1) If $x \in A$ and ||1-x|| < 1 then $x \in Inv(A)$
- (Q_2) If $x \in A$ and ||x|| < 1 then $\sum x^n$ converges in A
- $(Q_3) \quad r(x) = \lim_n \|x^n\|^{1/n} = \inf_n \|x^n\|^{1/n} \text{ for all } x \in A$
- $(Q_4) \quad \sup_{\|x\|=1} r(x) < \infty$
- (Q'_4) $r(x) \leq ||x||$ for all $x \in A$
- (Q_5) $\partial \operatorname{Inv}(A) \subset \operatorname{TDZ}(A)$
- (Q_6) Rad (A) is closed and A/Rad (A) is a Q-algebra
- (Q_7) $\sigma: A \to P(\mathbb{C}), x \mapsto \operatorname{Sp}(x)$ is upper semicontinuous
- (Q'_7) σ is upper semicontinuous at $0 \in A$
- (Q_8) D: $x \mapsto \text{diam}(Sp(x))$ is upper semicontinuous
- (Q_8') D is continuous at $0 \in A$.

Remarks on notation: 1. TDZ (A) in (Q₅) is the set of topological divisors of zero in A. Recall that x is in TDZ (A) if there is a sequence (w_n) in A with $||w_n|| = 1$ for all n, and such that $\lim_{n} w_n x = 0 = \lim_{n} x w_n$ (see [2], p. 12).

- 2. Rad (A) is defined as the intersection of all maximal left ideals in A (see [2], p. 124). Rad stands for *radical*.
- 3. If $x \in A$, then $\operatorname{Sp}(x) := \{\lambda \in \mathbb{C} : \lambda \, 1 x \notin \operatorname{Inv}(A)\}$ is the *spectrum* of x (in A). $r(x) := \sup\{|\lambda| : \lambda \in \operatorname{Sp}(x)\}$ is the *spectral radius* of x.
- 4. If K is a subset of A, ∂K is the boundary of K in A, i.e. $\partial K = \overline{K} \setminus \mathring{K}$, where \overline{K} is the closure of K and \mathring{K} is the set of its inner points.
- 5. Upper semicontinuous in (Q_7) means that, for each $x \in A$ and each open subset U of \mathbb{C} such that $\operatorname{Sp}(x) \subset U$, there is a $\delta > 0$ with $||y-x|| < \delta \Rightarrow \operatorname{Sp}(y) \subset U$.

Proof:

 $(Q) \Rightarrow (Q_{\text{FMI}})$: This is trivial since if Inv (A) is open then 1 is an inner point of Inv (A), that is, there exists a $\delta > 0$ with $\{y: ||1-y|| < \delta\} \subset \text{Inv}(A)$. Since $0 \notin \text{Inv}(A)$, clearly $\delta \le 1$.

 $(Q_{\text{FMI}}) \Rightarrow (Q_4)$: Let $\delta \in (0,1]$ be as in (Q_{FMI}) . We have $r(x) \leq \frac{\|x\|}{\delta}$ for all $x \in A$, hence $\sup_{\|x\|=1} r(x) \leq 1/\delta$.

 $(Q_4) \Rightarrow (Q_3)$: The formulas $r(a) \ge \lim_n \|a^n\|^{1/n} = \inf_n \|a^n\|^{1/n}$ and $\operatorname{Sp}(a) \ne \emptyset$ are true in all complex normed algebras (see [2], Prop. 2.8 and Th. 5.7). It remains to prove that $r(x) \le \lim_n \|x^n\|^{1/n}$ for all $x \in A$. Since $\operatorname{Sp}(q(x)) = q(\operatorname{Sp}(x))$ for all nonconstant poly-

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nomials q ([2], Prop. 5.5), we have, for $n \in \mathbb{N}$,

$$r(x)^n = r(x^n) \le M \cdot ||x^n||,$$

where $0 < M := \sup_{\|y\|=1} r(y) < \infty$. Now it follows immediately that

$$r(x) \le \lim_{n} M^{1/n} \|x^n\|^{1/n} = \lim_{n} \|x^n\|^{1/n}.$$

$$(Q_3) \Rightarrow (Q'_4): r(x) = \inf_n ||x^n||^{1/n} \le ||x||, \text{ for all } x \in A.$$

 $(Q'_4) \Rightarrow (Q_1)$: Let $x \in A$ and ||1-x|| < 1. Then r(1-x) < 1, that is, $1 \notin \text{Sp}(1-x)$, hence $x = 1 - (1-x) \in \text{Inv}(A)$.

 $(Q_1) \Rightarrow (Q_2)$: (see [3]). Let (Q_1) hold and let ||x|| < 1. Define $s_N := \sum_{n=0}^N x^n$ for all $N \ge 0$ ($x^0 := 1$). By (Q_1) , 1 - x is invertible. Let $y := (1 - x)^{-1}$. We have then

$$|| s_N - y || = || y (1 - x) s_N - y || \le || y || || (1 - x) s_N - 1 ||$$

$$= || y || || x^{N+1} || \le || y || || x ||^{N+1}.$$

Since ||x|| < 1, we get $\lim_{N} s_N = y$, that is, $\sum x^n$ converges.

 $(Q_2) \Rightarrow (Q_1)$: Let $x \in A$ and ||1-x|| < 1. It follows from (Q_2) that $\sum (1-x)^n$ converges to some $y \in A$. Now, since

$$\left\| 1 - x \sum_{n=0}^{N} (1 - x)^n \right\| = \left\| 1 + (1 - x) \sum_{n=0}^{N} (1 - x)^n - \sum_{n=0}^{N} (1 - x)^n \right\|$$
$$= \left\| (1 - x)^{N+1} \right\| \le \left\| 1 - x \right\|^{N+1},$$

we get x y = 1. Similarly, y x = 1 and thus x is invertible.

 $(Q_1) \Rightarrow (Q)$: Let x be invertible, and let $y \in A$ with $||x-y|| < 1/||x^{-1}||$. This implies

$$||1-x^{-1}y|| = ||x^{-1}(x-y)|| < 1$$
,

that is, $x^{-1}y$ is invertible, by (Q_1) . Let $w := (x^{-1}y)^{-1}$. It is clear that $(wx^{-1})y = 1$, and thus y is left invertible. Analogously we prove that yx^{-1} is invertible and, with $z := (yx^{-1})^{-1}$, we have $y(x^{-1}z) = 1$. Since y is left and right invertible, y must be invertible. This proves that Inv (A) is open.

For completeness' sake we prove

 $(Q) \Rightarrow (Q_P)$: Let $x \in A \cap \text{Inv}(A^*)$, $x^{-1} \in A^* \setminus A$, where A^* is the completion of A. If we had $x \in \partial \text{Inv}(A)$, there would exist $(x_n) \in \text{Inv}(A)^N$ such $\lim_n x_n = x$. Since $\lim_n x_n^{-1} = x^{-1}$

in A^* , we would have in particular that $M := \sup_{n} ||x_n^{-1}|| < \infty$. Taking n sufficiently

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large, we would get $M \cdot ||x - x_n|| < 1$ and $x = x_n (1 + x_n^{-1} (x - x_n)) \in \text{Inv}(A)$, by $(Q) \Leftrightarrow (Q_1)$: a contradiction. Hence $x \notin \partial \text{Inv}(A)$.

Since Inv (A^*) is open in A^* , we get thus a neighbourhood U of x in A^* such that $U \subset \text{Inv}(A^*)$ and $U \cap \text{Inv}(A) = \emptyset$. Since $a \mapsto a^{-1}$ is a homeomorphism in A^* ([2], Prop. 2.6), we have that U^{-1} is open in A^* and $U^{-1} \cap A = \emptyset$, which contradicts the denseness of A in A^* .

 $(Q) \Rightarrow (Q_5)$: Let $x \in A \cap \partial \operatorname{Inv}(A)$, $(x_n) \in \operatorname{Inv}(A)^{\mathbb{N}}$ such that $\lim_n x_n = x$. We claim that $\sup_n \|x_n^{-1}\| = \infty$. In fact, if we assume that $\|x_n^{-1}\| \le N < \infty$ for all n, we have

$$||x_m^{-1} - x_n^{-1}|| = ||x_m^{-1}(x_n - x_m)x_n^{-1}|| \le N^2 \cdot ||x_n - x_m||.$$

It follows that (x_n^{-1}) is a Cauchy sequence in A (say, with limit $y \in A^*$, A^* the completion of A). Then xy = yx = 1 by continuity of multiplication and thus $x \in \text{Inv}(A^*)$. By $(Q_P) \leftarrow (Q)$, $x \in \text{Inv}(A)$, which is a contradiction.

Without loss of generality let also $||x_n^{-1}|| \ge n$, for all n, and define $w_n := x_n^{-1}/||x_n^{-1}||$. It is now easy to see that $\lim_n x w_n = \lim_n w_n x = 0$, that is, $x \in TDZ(A)$.

- $(Q_5) \Rightarrow (Q)$: If A were not a Q-algebra, there would exist $x \in \text{Inv}(A) \cap \partial \text{Inv}(A)$. Since $x \in \text{Inv}(A)$, x cannot be in TDZ(A), contradicting (Q_5) .
- $(Q) \Rightarrow (Q_6)$: If A is a Q-algebra, then maximal left ideals are closed. This is an easy consequence of $J \subset A \setminus Inv(A)$ for every proper left ideal J. It is also an easy task to prove that A/I is a Q-algebra for every ideal I.
- $(Q_6) \Rightarrow (Q_3)$: We have that

Rad
$$(A) = \{x: 1-x \ y \in \text{Inv}(A) \text{ and } 1-y \ x \in \text{Inv}(A) \text{ for all } y \in A\}$$

([2], Prop. 24.16, Cor. 24.17). Using this result, we may follow Aupetit ([1], Lemme I,1.2) to obtain

$$\operatorname{Sp}(x) = \operatorname{Sp}(\hat{x})$$

for all $x \in A$, where \hat{x} denotes the class of x in A/Rad(A). Let $x \in A$. Since A/Rad(A) is a Q-algebra, $(Q) \Leftrightarrow (Q_3)$ gives

$$r(\hat{x}) = \lim_{n} \| \hat{x}^n \|^{1/n}$$

and thus

$$\lim_{n} \|x^{n}\|^{1/n} \le r(x) = \lim_{n} \|\hat{x}^{n}\|^{1/n} \le \lim_{n} \|x^{n}\|^{1/n},$$

which was to be proved (the first inequality follows from the general theorem already quoted in the proof of $(Q_4) \Rightarrow (Q_3)$).

 $(Q) \Rightarrow (Q_7)$: Let σ be not upper continuous at $x \in A$. Choose U open in \mathbb{C} such that $\operatorname{Sp}(x) \subset U$ and $(x_n) \in A^{\mathbb{N}}$, $(\alpha_n) \in \mathbb{C}^{\mathbb{N}}$ such that $\lim_n x_n = x$, $\alpha_n \in \operatorname{Sp}(x_n) \setminus U$. Since (by $(Q) \Leftrightarrow (Q'_4)$)

$$\sup_{n} |\alpha_{n}| \leq \sup_{n} r(x_{n}) \leq \sup_{n} ||x_{n}||,$$

we may assume that the x_n are chosen in such a way that (α_n) converges. Let $\alpha := \lim \alpha_n$.

Then $\alpha \notin U$ and since $\alpha_n 1 - x_n \notin \text{Inv}(A)$ for all n, and since $\alpha 1 - x = \lim_n \alpha_n 1 - x_n$, we have that $\alpha \notin \text{Sp}(x)$ and $\alpha 1 - x \in \text{Inv}(A) \cap \partial \text{Inv}(A)$, a contradiction with (Q).

$$(Q_7) \Rightarrow (Q_7') \Rightarrow (Q_8')$$
 and $(Q_7) \Rightarrow (Q_8) \Rightarrow (Q_8')$ are clear.

 $(Q_8') \Rightarrow (Q)$: Choose $\delta > 0$ such that $||x|| < \delta$ implies $\operatorname{Sp}(x) \subset U_{1/2}(0)$. It follows that $0 \notin \operatorname{Sp}(1-x) = 1 - \operatorname{Sp}(x)$, that is, $1-x \in \operatorname{Inv}(A)$. $(Q) \Leftrightarrow (Q_1)$ now does the rest.

Remarks: 1. As regards Palmer's characterization (Q_P) , the implication $(Q_P) \Rightarrow (Q)$ is very easy to prove: if $x \in A$ and ||1-x|| < 1, then $x \in \text{Inv}(A^*)$, since A^* is a Banach algebra, but this implies $x \in \text{Inv}(A)$ by (Q_P) .

- 2. I believe that our Theorem may sufficiently increase the popularity of normed Q-algebras. It is now clear that lots of elementary results about Banach algebras are true for Q-algebras, too: it is unfortunate that they are usually confusingly proved under completeness assumptions (see, for instance, [5], Chapter 18).
- 3. Our Theorem clearly has many applications. One may use the standard Banach-algebra-proofs to obtain, for instance, the following "Gelfand-Theorems":

Theorem (*): Commutative complex normed Q-algebras are exactly those A, for which there exist a compact space K and an isomorphism ϕ of A/Rad (A) onto a full subalgebra of C(K), which is separating in C(K) and contains 1_K .

Theorem ():** Commutative Q^* -algebras (defined analogously to C^* -algebras) are the full dense subalgebras of C(K) which are separating and contain 1_K , for a certain compact space K.

(Recall that a subalgebra B of A is full if B contains the unity of A and if, whenever $b \in B$ has an inverse b^{-1} in A, b^{-1} is in B. A subalgebra of C(K) is separating if, given points p and q in K, there is an f in A with $f(p) \neq f(q)$.) It is a very useful exercise to prove these theorems!

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