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Some characterizations of complex normed Q -algebras

A complex normed algebra A with unit 1 is a normed vector space over \mathbb{C} where a multiplication $A \times A \rightarrow A$ is defined such that

$$\|x \cdot y\| \leq \|x\| \|y\|, \quad \text{for all } x, y \in A,$$

where \cdot denotes the multiplication and $\|\cdot\|$ the norm on A . With respect to this multiplication the unit 1 of A has of course the property $1 \cdot x = x \cdot 1 = x$ for every $x \in A$.

The normed algebras we usually meet in the applications and in introductory functional analysis courses are the so-called *Banach algebras*. These are normed algebras such that the norm induces a *complete* topology (i.e. one such that Cauchy sequences converge).

Now, every second year mathematics student knows that the set of all invertible elements of a Banach algebra A is *open* ($x \in A$ is *invertible* if there is a $y \in A$ such that $x \cdot y = y \cdot x = 1$). In such case we write $y =: x^{-1}$. The set of invertibles of A is denoted by $\text{Inv}(A)$. The point is that the converse of this statement is false: just take a look at the algebra $R(D)$ of all complex rational functions defined on the closed unit disk of \mathbb{C} , endowed with the norm $\|q\| := \sup_{z \in \mathbb{C}} |q(z)|$. $q \in R(D)$ is invertible if and only if it has

no zeros in D , and thus $\text{Inv}(R(D))$ is open in $R(D)$, by the Maximum Principle. On the other hand, $R(D)$ is clearly no Banach algebra since there are analytic functions on D which are not rational (e.g. $\sin(z)$!).

Since the condition that $\text{Inv}(A)$ be open has a well-mixed topological and algebraic nature, it seems interesting to define *Q -algebras* (or *open algebras*, as they are sometimes called) as those algebras (with unit!) which satisfy it. Of course, all Banach algebras are Q -algebras.

Our purpose is to show that almost all fundamental properties of Banach algebras are shared by the larger class of normed Q -algebras. Quite surprisingly, it turns out that some of these properties do actually characterize the normed Q -algebras among the normed algebras with unit.

In the following A will be a complex normed algebra with unit 1 and norm $\|\cdot\|$.

Fuster and Marquina [3] have proved the equivalence of the statements

(Q) A is a Q -algebra

(Q_{FM1}) $\exists \delta \in (0, 1] : x \in A$ and $\|1 - x\| < \delta$ imply $x \in \text{Inv}(A)$

(Q_{FM2}) $\exists \delta \in (0, 1] : x \in A$ and $\|x\| < \delta$ imply that $\sum x^n$ converges in A .

In an unpublished paper [4] Th. W. Palmer has given a further characterization:

(Q_P) A is inverse-closed in its completion, that is, if A^* is the completion of A , then $\text{Inv}(A^*) \cap A \subset \text{Inv}(A)$.

Let's now state our theorem:

Theorem. *Let A be a complex normed algebra with unit 1. Then the following conditions are equivalent:*

- (Q) A is a Q -algebra
- (Q₁) If $x \in A$ and $\|1 - x\| < 1$ then $x \in \text{Inv}(A)$
- (Q₂) If $x \in A$ and $\|x\| < 1$ then $\sum x^n$ converges in A
- (Q₃) $r(x) = \lim_n \|x^n\|^{1/n} = \inf_n \|x^n\|^{1/n}$ for all $x \in A$
- (Q₄) $\sup_{\|x\|=1} r(x) < \infty$
- (Q_{4'}) $r(x) \leq \|x\|$ for all $x \in A$
- (Q₅) $\partial \text{Inv}(A) \subset \text{TDZ}(A)$
- (Q₆) $\text{Rad}(A)$ is closed and $A/\text{Rad}(A)$ is a Q -algebra
- (Q₇) $\sigma: A \rightarrow P(\mathbb{C}), x \mapsto \text{Sp}(x)$ is upper semicontinuous
- (Q_{7'}) σ is upper semicontinuous at $0 \in A$
- (Q₈) $D: x \mapsto \text{diam}(\text{Sp}(x))$ is upper semicontinuous
- (Q_{8'}) D is continuous at $0 \in A$.

Remarks on notation: 1. $\text{TDZ}(A)$ in (Q₅) is the set of *topological divisors of zero* in A . Recall that x is in $\text{TDZ}(A)$ if there is a sequence (w_n) in A with $\|w_n\| = 1$ for all n , and such that $\lim_n w_n x = 0 = \lim_n x w_n$ (see [2], p. 12).

2. $\text{Rad}(A)$ is defined as the intersection of all maximal left ideals in A (see [2], p. 124). Rad stands for *radical*.

3. If $x \in A$, then $\text{Sp}(x) := \{\lambda \in \mathbb{C} : \lambda 1 - x \notin \text{Inv}(A)\}$ is the *spectrum* of x (in A). $r(x) := \sup\{|\lambda| : \lambda \in \text{Sp}(x)\}$ is the *spectral radius* of x .

4. If K is a subset of A , ∂K is the *boundary* of K in A , i.e. $\partial K = \bar{K} \setminus \mathring{K}$, where \bar{K} is the closure of K and \mathring{K} is the set of its inner points.

5. *Upper semicontinuous* in (Q₇) means that, for each $x \in A$ and each open subset U of \mathbb{C} such that $\text{Sp}(x) \subset U$, there is a $\delta > 0$ with $\|y - x\| < \delta \Rightarrow \text{Sp}(y) \subset U$.

Proof:

(Q) \Rightarrow (Q_{FMI}): This is trivial since if $\text{Inv}(A)$ is open then 1 is an inner point of $\text{Inv}(A)$, that is, there exists a $\delta > 0$ with $\{y : \|1 - y\| < \delta\} \subset \text{Inv}(A)$. Since $0 \notin \text{Inv}(A)$, clearly $\delta \leq 1$.

(Q_{FMI}) \Rightarrow (Q₄): Let $\delta \in (0, 1]$ be as in (Q_{FMI}). We have $r(x) \leq \frac{\|x\|}{\delta}$ for all $x \in A$, hence $\sup_{\|x\|=1} r(x) \leq 1/\delta$.

(Q₄) \Rightarrow (Q₃): The formulas $r(a) \leq \lim_n \|a^n\|^{1/n} = \inf_n \|a^n\|^{1/n}$ and $\text{Sp}(a) \neq \emptyset$ are true in all complex normed algebras (see [2], Prop. 2.8 and Th. 5.7). It remains to prove that $r(x) \leq \lim_n \|x^n\|^{1/n}$ for all $x \in A$. Since $\text{Sp}(q(x)) = q(\text{Sp}(x))$ for all nonconstant poly-

nomials q ([2], Prop. 5.5), we have, for $n \in \mathbb{N}$,

$$r(x)^n = r(x^n) \leq M \cdot \|x^n\|,$$

where $0 < M := \sup_{\|y\|=1} r(y) < \infty$. Now it follows immediately that

$$r(x) \leq \lim_n M^{1/n} \|x^n\|^{1/n} = \lim_n \|x^n\|^{1/n}.$$

$(Q_3) \Rightarrow (Q'_4)$: $r(x) = \inf_n \|x^n\|^{1/n} \leq \|x\|$, for all $x \in A$.

$(Q'_4) \Rightarrow (Q_1)$: Let $x \in A$ and $\|1-x\| < 1$. Then $r(1-x) < 1$, that is, $1 \notin \text{Sp}(1-x)$, hence $x = 1 - (1-x) \in \text{Inv}(A)$.

$(Q_1) \Rightarrow (Q_2)$: (see [3]). Let (Q_1) hold and let $\|x\| < 1$. Define $s_N := \sum_{n=0}^N x^n$ for all $N \geq 0$ ($x^0 := 1$). By (Q_1) , $1-x$ is invertible. Let $y := (1-x)^{-1}$. We have then

$$\begin{aligned} \|s_N - y\| &= \|y(1-x)s_N - y\| \leq \|y\| \|(1-x)s_N - 1\| \\ &= \|y\| \|x^{N+1}\| \leq \|y\| \|x\|^{N+1}. \end{aligned}$$

Since $\|x\| < 1$, we get $\lim_N s_N = y$, that is, $\sum x^n$ converges.

$(Q_2) \Rightarrow (Q_1)$: Let $x \in A$ and $\|1-x\| < 1$. It follows from (Q_2) that $\sum (1-x)^n$ converges to some $y \in A$. Now, since

$$\begin{aligned} \left\| 1 - x \sum_{n=0}^N (1-x)^n \right\| &= \left\| 1 + (1-x) \sum_{n=0}^N (1-x)^n - \sum_{n=0}^N (1-x)^n \right\| \\ &= \|(1-x)^{N+1}\| \leq \|1-x\|^{N+1}, \end{aligned}$$

we get $xy = 1$. Similarly, $yx = 1$ and thus x is invertible.

$(Q_1) \Rightarrow (Q)$: Let x be invertible, and let $y \in A$ with $\|x-y\| < 1/\|x^{-1}\|$. This implies

$$\|1 - x^{-1}y\| = \|x^{-1}(x-y)\| < 1,$$

that is, $x^{-1}y$ is invertible, by (Q_1) . Let $w := (x^{-1}y)^{-1}$. It is clear that $(wx^{-1})y = 1$, and thus y is left invertible. Analogously we prove that yx^{-1} is invertible and, with $z := (yx^{-1})^{-1}$, we have $y(x^{-1}z) = 1$. Since y is left and right invertible, y must be invertible. This proves that $\text{Inv}(A)$ is open.

For completeness' sake we prove

$(Q) \Rightarrow (Q_P)$: Let $x \in A \cap \text{Inv}(A^*)$, $x^{-1} \in A^* \setminus A$, where A^* is the completion of A . If we had $x \in \partial \text{Inv}(A)$, there would exist $(x_n) \in \text{Inv}(A)^{\mathbb{N}}$ such $\lim_n x_n = x$. Since $\lim_n x_n^{-1} = x^{-1}$ in A^* , we would have in particular that $M := \sup_n \|x_n^{-1}\| < \infty$. Taking n sufficiently

large, we would get $M \cdot \|x - x_n\| < 1$ and $x = x_n(1 + x_n^{-1}(x - x_n)) \in \text{Inv}(A)$, by $(Q) \Leftrightarrow (Q_1)$: a contradiction. Hence $x \notin \partial \text{Inv}(A)$.

Since $\text{Inv}(A^*)$ is open in A^* , we get thus a neighbourhood U of x in A^* such that $U \subset \text{Inv}(A^*)$ and $U \cap \text{Inv}(A) = \emptyset$. Since $a \mapsto a^{-1}$ is a homeomorphism in A^* ([2], Prop. 2.6), we have that U^{-1} is open in A^* and $U^{-1} \cap A = \emptyset$, which contradicts the denseness of A in A^* .

$(Q) \Rightarrow (Q_5)$: Let $x \in A \cap \partial \text{Inv}(A)$, $(x_n) \in \text{Inv}(A)^{\mathbb{N}}$ such that $\lim_n x_n = x$. We claim that $\sup_n \|x_n^{-1}\| = \infty$. In fact, if we assume that $\|x_n^{-1}\| \leq N < \infty$ for all n , we have

$$\|x_m^{-1} - x_n^{-1}\| = \|x_m^{-1}(x_n - x_m)x_n^{-1}\| \leq N^2 \cdot \|x_n - x_m\|.$$

It follows that (x_n^{-1}) is a Cauchy sequence in A (say, with limit $y \in A^*$, A^* the completion of A). Then $xy = yx = 1$ by continuity of multiplication and thus $x \in \text{Inv}(A^*)$. By $(Q_p) \Leftrightarrow (Q)$, $x \in \text{Inv}(A)$, which is a contradiction.

Without loss of generality let also $\|x_n^{-1}\| \geq n$, for all n , and define $w_n := x_n^{-1} / \|x_n^{-1}\|$. It is now easy to see that $\lim_n x w_n = \lim_n w_n x = 0$, that is, $x \in \text{TDZ}(A)$.

$(Q_5) \Rightarrow (Q)$: If A were not a Q -algebra, there would exist $x \in \text{Inv}(A) \cap \partial \text{Inv}(A)$. Since $x \in \text{Inv}(A)$, x cannot be in $\text{TDZ}(A)$, contradicting (Q_5) .

$(Q) \Rightarrow (Q_6)$: If A is a Q -algebra, then maximal left ideals are closed. This is an easy consequence of $J \subset A \setminus \text{Inv}(A)$ for every proper left ideal J . It is also an easy task to prove that A/I is a Q -algebra for every ideal I .

$(Q_6) \Rightarrow (Q_3)$: We have that

$$\text{Rad}(A) = \{x: 1 - xy \in \text{Inv}(A) \text{ and } 1 - yx \in \text{Inv}(A) \text{ for all } y \in A\}$$

([2], Prop. 24.16, Cor. 24.17). Using this result, we may follow Aupetit ([1], Lemme I,1.2) to obtain

$$\text{Sp}(x) = \text{Sp}(\hat{x})$$

for all $x \in A$, where \hat{x} denotes the class of x in $A/\text{Rad}(A)$. Let $x \in A$. Since $A/\text{Rad}(A)$ is a Q -algebra, $(Q) \Leftrightarrow (Q_3)$ gives

$$r(\hat{x}) = \lim_n \|\hat{x}^n\|^{1/n}$$

and thus

$$\lim_n \|x^n\|^{1/n} \leq r(x) = \lim_n \|\hat{x}^n\|^{1/n} \leq \lim_n \|x^n\|^{1/n},$$

which was to be proved (the first inequality follows from the general theorem already quoted in the proof of $(Q_4) \Rightarrow (Q_3)$).

$(Q) \Rightarrow (Q_7)$: Let σ be not upper continuous at $x \in A$. Choose U open in \mathbb{C} such that $\text{Sp}(x) \subset U$ and $(x_n) \in A^{\mathbb{N}}$, $(\alpha_n) \in \mathbb{C}^{\mathbb{N}}$ such that $\lim_n x_n = x$, $\alpha_n \in \text{Sp}(x_n) \setminus U$. Since (by $(Q) \Leftrightarrow (Q_4)$)

$$\sup_n |\alpha_n| \leq \sup_n r(x_n) \leq \sup_n \|x_n\|,$$

we may assume that the x_n are chosen in such a way that (α_n) converges. Let $\alpha := \lim_n \alpha_n$.

Then $\alpha \notin U$ and since $\alpha_n 1 - x_n \notin \text{Inv}(A)$ for all n , and since $\alpha 1 - x = \lim_n \alpha_n 1 - x_n$, we have that $\alpha \notin \text{Sp}(x)$ and $\alpha 1 - x \in \text{Inv}(A) \cap \partial \text{Inv}(A)$, a contradiction with (Q) .

$(Q_7) \Rightarrow (Q'_7) \Rightarrow (Q'_8)$ and $(Q_7) \Rightarrow (Q_8) \Rightarrow (Q'_8)$ are clear.

$(Q'_8) \Rightarrow (Q)$: Choose $\delta > 0$ such that $\|x\| < \delta$ implies $\text{Sp}(x) \subset U_{1/2}(0)$. It follows that $0 \notin \text{Sp}(1-x) = 1 - \text{Sp}(x)$, that is, $1-x \in \text{Inv}(A)$. $(Q) \Leftrightarrow (Q_1)$ now does the rest.

Remarks: 1. As regards Palmer's characterization (Q_P) , the implication $(Q_P) \Rightarrow (Q)$ is very easy to prove: if $x \in A$ and $\|1-x\| < 1$, then $x \in \text{Inv}(A^*)$, since A^* is a Banach algebra, but this implies $x \in \text{Inv}(A)$ by (Q_P) .

2. I believe that our Theorem may sufficiently increase the popularity of normed Q -algebras. It is now clear that lots of elementary results about Banach algebras are true for Q -algebras, too: it is unfortunate that they are usually confusingly proved under completeness assumptions (see, for instance, [5], Chapter 18).

3. Our Theorem clearly has many applications. One may use the standard Banach-algebra-proofs to obtain, for instance, the following "Gelfand-Theorems":

Theorem (*): *Commutative complex normed Q -algebras are exactly those A , for which there exist a compact space K and an isomorphism ϕ of $A/\text{Rad}(A)$ onto a full subalgebra of $C(K)$, which is separating in $C(K)$ and contains 1_K .*

Theorem ()**: *Commutative Q^* -algebras (defined analogously to C^* -algebras) are the full dense subalgebras of $C(K)$ which are separating and contain 1_K , for a certain compact space K .*

(Recall that a subalgebra B of A is *full* if B contains the unity of A and if, whenever $b \in B$ has an inverse b^{-1} in A , b^{-1} is in B . A subalgebra of $C(K)$ is *separating* if, given points p and q in K , there is an f in A with $f(p) \neq f(q)$.) It is a very useful exercise to prove these theorems!

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REFERENCES

- 1 B. Aupetit: Propriétés spectrales des algèbres de Banach, LNM 735. Springer-Verlag, Berlin 1979.
- 2 F. F. Bonsall and J. Duncan: Complete normed algebras. Springer-Verlag, Berlin 1973.
- 3 R. Fuster and A. Marquina: Geometric series in incomplete normed algebras. Am. Math. Monthly 91, 49–51 (1984).
- 4 Th. W. Palmer: Spectral algebras, subalgebras and pseudo-norms. To appear.
- 5 W. Rudin: Real and complex analysis. McGraw-Hill, New York 1974.