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An elementary proof of the Theorem of Beckman and Quarles

1. I have been asked by colleagues to write down that proof of the fundamental and classical Theorem of Beckman, Quarles [1] that I have presented in a beginners course on Geometric Transformations for students already familiar with the basic methods of Linear Algebra. The proof in question, which is already sketched in a more general context in [2], is a mixture of ideas of Beckman, Quarles [1], Schröder [5], Benz [2] up to some new details. In this connection we also refer to Parhomenko and Modenov [4] and to their proof of the Theorem in question.

Let \mathbb{R}^n ($1 < n < \infty$) be equipped with the usual scalar product

$$a \cdot b := \sum_{i=1}^n \alpha_i \beta_i$$

for $a = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $b = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$.
Then

$$\|a - b\| := \sqrt{(a - b)^2}$$

is called the distance of $a, b \in \mathbb{R}^n$.

Theorem of Beckman and Quarles: Suppose $k > 0$ to be a fixed real number and suppose f to be a mapping of \mathbb{R}^n ($1 < n < \infty$) into itself such that

$$\|p - q\| = k \text{ implies } \|f(p) - f(q)\| = k$$

for all $p, q \in \mathbb{R}^n$. Then f is an isometry of \mathbb{R}^n and hence a bijective linear mapping up to a translation.

In section 2 we shall collect some simple facts which are useful later on. Those elementary facts could be presented in a course far ahead the proof of the theorem in question, possibly in the form of exercises for the students.

The proof itself will be given in sections 3 and 4. It might be noticed that the original theorem in [1] was formulated for multivalued transformations f . This is however no substantial generalization as was pointed out in [3] in the case of Lorentz transformations of \mathbb{R}^n .

2. Throughout this note exactly the elements of \mathbb{R}^n ($1 < n < \infty$) are called points.

1) Suppose that a, m, b are points such that

$$\|m - a\| = \|b - m\| = \frac{1}{2} \|b - a\|.$$

Then $m = \frac{1}{2}(a + b)$.

Proof: Putting $\varrho := \|m - a\|$, $a' := m - a$, $b' := b - m$ we have $(b - a)^2 + (a' - b')^2 = (a' + b')^2 + (a' - b')^2 = 4\varrho^2$ and hence $(a' - b')^2 = 0$.

2) A set of n distinct points of \mathbb{R}^n which are pairwise of distance $\beta > 0$ will be called a β -set. Suppose that α, β are positive real numbers with

$$\gamma(\alpha, \beta) := 4\alpha^2 - 2\beta^2 \left(1 - \frac{1}{n}\right) > 0$$

and suppose that P is a β -set. Then there exist exactly two distinct points in \mathbb{R}^n which have distance α from all $p \in P$. Those two points will be called the α -associated points of P . Their distance is $\sqrt{\gamma(\alpha, \beta)}$.

Proof: a) Let $P = \{p_1, \dots, p_n\}$ be a β -set. Then for $i, j \in \{1, 2, \dots, n-1\}$ with $i \neq j$ we have

$$(p_i - p_n)(p_j - p_n) = \frac{1}{2}\beta^2,$$

because of $\beta^2 = (p_i - p_j)^2 = ((p_i - p_n) - (p_j - p_n))^2$. Define $\lambda_r := \frac{\beta}{\sqrt{2r(r+1)}}$ for $r = 1, 2, \dots$ and e_1, \dots, e_{n-1} by $(1+s)\lambda_s e_s := (p_s - p_n) - \sum_{r=1}^{s-1} \lambda_r e_r$ for $s = 1, \dots, n-1$.

Obviously, $e_i^2 = 1$. We now prove

$$e_i e_j = \begin{cases} 1 & \text{for } i = j \leq n-1 \\ 0 & \text{for } i < j \leq n-1 \end{cases}$$

by induction along the sequence

$$(1, 1), (1, 2), (2, 2), (1, 3), (2, 3), (3, 3), \dots, (n-1, n-1) \quad \text{for } (i, j):$$

Step $(i, i) \rightarrow (1, i+1)$: Here we have

$$\begin{aligned} \frac{1}{2}\beta^2 &= (p_1 - p_n)(p_{i+1} - p_n) = 2\lambda_1 e_1 \left(\sum_{r=1}^i \lambda_r e_r + (2+i)\lambda_{i+1} e_{i+1} \right) \\ &= 2\lambda_1^2 + 2(2+i)\lambda_1 \lambda_{i+1} e_1 e_{i+1}, \end{aligned}$$

and hence $e_1 e_{i+1} = 0$, because of $\frac{1}{2}\beta^2 = 2\lambda_1^2$.

Step $(i-1, j) \rightarrow (i, j)$ in case $i < j$: Here we have

$$\begin{aligned} \frac{1}{2}\beta^2 &= (p_i - p_n)(p_j - p_n) = \left(\sum_{r=1}^{i-1} \lambda_r e_r + (1+i)\lambda_i e_i \right) \left(\sum_{r=1}^{j-1} \lambda_r e_r + (1+j)\lambda_j e_j \right) \\ &= \sum_{r=1}^{i-1} \lambda_r^2 + (1+i)\lambda_i^2 + (1+i)(1+j)\lambda_i \lambda_j e_i e_j, \end{aligned}$$

and hence $e_i e_j = 0$, because of $\frac{1}{2}\beta^2 = \sum_{r=1}^{i-1} \lambda_r^2 + (1+i)\lambda_i^2$ by observing

$$\lambda_r^2 = \frac{\beta^2}{2} \left(\frac{1}{r} - \frac{1}{r+1} \right).$$

Step $(i-1, i) \rightarrow (i, i)$: We finally have

$$\beta^2 = (p_i - p_n)^2 = \left(\sum_{r=1}^{i-1} \lambda_r e_r + (1+i) \lambda_i e_i \right)^2 = \sum_{r=1}^{i-1} \lambda_r^2 + (1+i)^2 \lambda_i^2 e_i^2,$$

and hence $e_i^2 = 1$.

b) Suppose now that $q \in \mathbb{R}^n$ has distance α from all $p_s \in P$. This implies

$$(q - p_n)(p_s - p_n) = \frac{1}{2} \beta^2 \quad \text{for all } s = 1, \dots, n-1,$$

because of $\alpha^2 = (q - p_s)^2 = ((q - p_n) - (p_s - p_n))^2$.

Put $q - p_n := \sum_{r=1}^n \mu_r e_r$, $\mu_r \in \mathbb{R}$, by extending $\{e_1, \dots, e_{n-1}\}$ of part a) to an orthonormal basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n . We get the equation

$$\frac{1}{2} \beta^2 = (q - p_n)(p_s - p_n) = \sum_{r=1}^{s-1} \mu_r \lambda_r + (1+s) \mu_s \lambda_s \quad \text{for } s = 1, \dots, n-1.$$

The case $s = 1$ leads to $\mu_1 = \lambda_1$, and having already $\mu_i = \lambda_i$ for $i \in \{1, \dots, s-1\}$, $s < n$, we also get $\mu_s = \lambda_s$ by comparing the equation above with

$$\frac{1}{2} \beta^2 = \sum_{r=1}^{s-1} \lambda_r^2 + (1+s) \lambda_s^2.$$

Hence $q - p_n = \sum_{r=1}^{n-1} \lambda_r e_r + \mu_n e_n$. Now $(q - p_n)^2 = \alpha^2$ leads to

$$\mu_n^2 = \alpha^2 - \sum_{r=1}^{n-1} \lambda_r^2 = \alpha^2 - \frac{\beta^2}{2} \left(1 - \frac{1}{n} \right) = \frac{1}{4} \gamma(\alpha, \beta).$$

There are exactly two solutions q , namely the points

$$q_i = p_n + \sum_{r=1}^{n-1} \lambda_r e_r \pm \frac{1}{2} \sqrt{\gamma(\alpha, \beta)} \cdot e_n, \quad i = 1, 2,$$

which are in fact of distance α from all $p \in P$. Obviously, $(q_1 - q_2)^2 = \gamma(\alpha, \beta)$.

3) Again suppose that α, β are positive real numbers with $\gamma(\alpha, \beta) > 0$. Let x, y be points of distance $\sqrt{\gamma(\alpha, \beta)}$. Then there exists a β -set P such that x, y are the α -associated points of P .

Proof: Define $e_n := \frac{y-x}{\sqrt{\gamma(\alpha, \beta)}}$ and extend $\{e_n\}$ to an orthonormal basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n . If p_n is an arbitrary point of \mathbb{R}^n , then $P = \{p_1, \dots, p_n\}$ with

$$p_s - p_n := \sum_{r=1}^{s-1} \lambda_r e_r + (1+s) \lambda_s e_s \quad \text{for } s = 1, \dots, n-1$$

is a β -set by using the earlier defined λ_r . If we now take the special point

$$p_n := \frac{x+y}{2} - \sum_{r=1}^{n-1} \lambda_r e_r,$$

then the α -associated points of P are given by (see part b) of 2))

$$q_i = p_n + \sum_{r=1}^{n-1} \lambda_r e_r + \frac{1}{2} \sqrt{\gamma(\alpha, \beta)} e_n = \frac{x+y}{2} \pm \frac{y-x}{2} = \begin{cases} y \\ x \end{cases}.$$

3. Proposition: Let $\varrho > 0$ be a fixed real number and let $N > 2$ be a fixed integer. Suppose that $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ($1 < n < \infty$) is a mapping such that

- $\alpha)$ $\|x-y\| = \varrho$ implies $\|f(x) - f(y)\| \leq \varrho$,
- $\beta)$ $\|x-y\| = N\varrho$ implies $\|f(x) - f(y)\| = N\varrho$

for all $x, y \in \mathbb{R}^n$. Then $\|x-y\| = \|f(x) - f(y)\|$ holds true for all $x, y \in \mathbb{R}^n$.

Proof: a) Distances ϱ and 2ϱ are preserved under f : Having points x, y with $\|x-y\| = \varrho$ define $z := 2y-x$ and having points x, z with $\|x-z\| = 2\varrho$ define $y := \frac{1}{2}(x+z)$. Put $p_\lambda := x + \frac{\lambda}{2}(z-x)$ for $\lambda = 0, 1, \dots, N$. Observe $\|f(p_0) - f(p_N)\| = N\varrho$ and $\|f(p_\lambda) - f(p_{\lambda+1})\| \leq \varrho$ for $\lambda = 0, 1, \dots, N-1$ because of $\|p_0 - p_N\| = N\varrho$ and $\|p_\lambda - p_{\lambda+1}\| = \varrho$. The triangle inequality yields

$$\begin{aligned} N\varrho = \|f(p_0) - f(p_N)\| &\leq \|f(p_0) - f(p_2)\| + \sum_{\lambda=2}^{N-1} \|f(p_\lambda) - f(p_{\lambda+1})\| \leq \\ &\leq \sum_{\lambda=0}^{N-1} \|f(p_\lambda) - f(p_{\lambda+1})\| \leq N\varrho \end{aligned}$$

and hence $\|f(p_\lambda) - f(p_{\lambda+1})\| = \varrho$ ($\lambda = 0, 1, \dots, N-1$) and

$$\|f(p_0) - f(p_2)\| = \|f(p_0) - f(p_1)\| + \|f(p_1) - f(p_2)\|.$$

Because of $p_0 = x, p_1 = y, p_2 = z$ we thus have

$$\|f(x) - f(z)\| = 2\varrho \quad \text{and} \quad \|f(x) - f(y)\| = \varrho.$$

b) Suppose that $\|x-y\| = \varrho$ for $x, y \in \mathbb{R}^n$. Then

$$f(x + \lambda(y-x)) = f(x) + \lambda(f(y) - f(x)) \tag{1}$$

holds true for all $\lambda = 0, 1, 2, \dots$: Put $p_\lambda := x + \lambda(y-x)$ for $\lambda = 0, 1, 2, \dots$ and observe

$$\|p_\lambda - p_{\lambda-1}\| = \varrho = \|p_{\lambda+1} - p_\lambda\| \quad \text{and} \quad \|p_{\lambda+1} - p_{\lambda-1}\| = 2\varrho$$

for $\lambda = 1, 2, \dots$. Since distances ϱ and 2ϱ are preserved we get

$$\varrho = \|f(p_\lambda) - f(p_{\lambda-1})\| = \|f(p_{\lambda+1}) - f(p_\lambda)\| = \frac{1}{2} \|f(p_{\lambda+1}) - f(p_{\lambda-1})\|$$

and hence (compare 1) in section 2) $f(p_\lambda) = \frac{1}{2}[f(p_{\lambda-1}) + f(p_{\lambda+1})]$. This proves (1) by induction since (1) is trivial in cases $\lambda = 0$ and $\lambda = 1$.

c) Let λ, μ be positive integers and suppose that $\|x - y\| = \frac{\lambda\varrho}{\mu}$ for $x, y \in \mathbb{R}^n$. Then $\|f(x) - f(y)\| = \frac{\lambda\varrho}{\mu}$ holds true: Because of $n > 1$ and $2\lambda\varrho > \|x - y\|$ there exists a point $z \in \mathbb{R}^n$ with $\|z - x\| = \lambda\varrho = \|z - y\|$. With such a fixed z define a, b by

$$x = z + \lambda(a - z), \quad y = z + \lambda(b - z) \quad (2)$$

and put

$$x' := z + \mu(a - z), \quad y' := z + \mu(b - z). \quad (3)$$

Since $\|a - z\| = \varrho = \|b - z\|$ we hence have the corresponding formulas to (2), (3) for the images because of b). Now

$$\|x' - y'\| = \varrho = \|f(x') - f(y')\| = \mu \|f(a) - f(b)\|$$

and

$$\|f(x) - f(y)\| = \lambda \|f(a) - f(b)\| \quad \text{imply} \quad \|f(x) - f(y)\| = \frac{\lambda\varrho}{\mu}.$$

d) Let r, s be positive rational numbers and let x, y be points such that $r\varrho < \|x - y\| < s\varrho$. Then $r\varrho \leq \|f(x) - f(y)\| \leq s\varrho$: Since $n > 1$ and $s\varrho > \|x - y\|$ there exists a point z with $\|z - x\| = \frac{s\varrho}{2} = \|z - y\|$. Now c) implies $\|f(z) - f(x)\| = \frac{s\varrho}{2} = \|f(z) - f(y)\|$ and hence $\|f(x) - f(y)\| \leq \|f(x) - f(z)\| + \|f(z) - f(y)\| = s\varrho$.

Put $w := x + \frac{s\varrho}{\|x - y\|}(y - x)$ and observe $\|w - x\| = s\varrho$ and

$$\|w - y\| = \left(\frac{s\varrho}{\|x - y\|} - 1 \right) \|y - x\| = s\varrho - \|y - x\| < (s - r)\varrho.$$

Hence $\|f(w) - f(x)\| = s\varrho$ by c) and $\|f(w) - f(y)\| \leq (s - r)\varrho$ by the already proved part of d). This implies

$$\|f(x) - f(y)\| \geq \|f(x) - f(w)\| - \|f(y) - f(w)\| \geq s\varrho - (s - r)\varrho = r\varrho.$$

4. Throughout this section let $k > 0$ be a fixed real number and f be a mapping of \mathbb{R}^n ($1 < n < \infty$) into itself such that distance k is preserved under f , i.e. $\|x - y\| = k$ implies $\|f(x) - f(y)\| = k$ for all $x, y \in \mathbb{R}^n$.

Lemma: Suppose that α, β are positive real numbers such that $\gamma(\alpha, \beta) > 0$ (compare section 2). Suppose moreover that f preserves distances α and β and that x, y are points with $\|x - y\| = \varepsilon := \sqrt{\gamma(\alpha, \beta)}$. Then $\|f(x) - f(y)\| \in \{0, \varepsilon\}$ and in case $2\varepsilon > \alpha$ we even have $\|f(x) - f(y)\| = \varepsilon$.

Proof: This is trivial for $\varepsilon = \alpha$ since distance α is preserved. So assume $\varepsilon \neq \alpha$. Let P be a β -set such that x, y are the α -associated points of P (compare 3) of section 2). It is $P' := f(P)$ also a β -set since distance β is preserved. If we denote the α -associated points of P' by x', y' we get $f(x), f(y) \in \{x', y'\}$ since distance α is also preserved under f and since the α -associated points of P' are uniquely determined. This implies $\|f(x) - f(y)\| \in \{0, \|x' - y'\|\} = \{0, \varepsilon\}$ according to 2) in section 2. Assume now $2\varepsilon > \alpha$. We have to show that $f(x) \neq f(y)$. Assume $f(x) = f(y)$ and take a $z \in \mathbb{R}^n$ with $\|z - x\| = \varepsilon$ and $\|y - z\| = \alpha$ which exists since $n > 1$ and $2\varepsilon > \alpha$. The already proved part of the lemma yields $\|f(x) - f(z)\| \in \{0, \varepsilon\}$, i.e. $\|f(y) - f(z)\| \in \{0, \varepsilon\}$ because of $f(x) = f(y)$. Hence $\alpha = \|y - z\| = \|f(y) - f(z)\| \in \{0, \varepsilon\}$. This contradicts $\varepsilon \neq \alpha > 0$.

We note the following three consequences of our Lemma:

- a) Putting $\alpha = k = \beta$ we realize that distance $\sqrt{\gamma(\alpha, \beta)} = k \sqrt{2 \left(1 + \frac{1}{n}\right)}$ is preserved.
- b) Putting $\alpha = \beta = k \sqrt{2 \left(1 + \frac{1}{n}\right)}$ we realize that distance $\sqrt{\gamma(\alpha, \beta)} = (n+1) \cdot \frac{2k}{n}$ is preserved.
- c) Put $\alpha = k$ and $\beta = k \sqrt{2 \left(1 + \frac{1}{n}\right)}$. Then $\|x - y\| = \sqrt{\gamma(\alpha, \beta)} = \frac{2k}{n}$ implies $\|f(x) - f(y)\| \in \left\{0, \frac{2k}{n}\right\}$, i.e. $\|f(x) - f(y)\| \leq \frac{2k}{n}$ for all $x, y \in \mathbb{R}^n$.

If we now take $\varrho := \frac{2k}{n}$ in the Proposition of section 3 and $N := n + 1$ we realize that f is an isometry according to c), b) and $n > 1$.

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