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On a diophantine equation $(x^2 - 1)(y^2 - 1) = (z^2 - 1)^2$

1. Schinzel and Sierpiński [4] gave the complete solution of the equation

$$(x^2 - 1)(y^2 - 1) = \left(\left(\frac{y-x}{2} \right)^2 - 1 \right)^2 \quad (1)$$

in positive integers $x, y, x < y$. Equation (1) is a special case of the equation

$$(x^2 - 1)(y^2 - 1) = (z^2 - 1)^2. \quad (2)$$

The only known solution of (2) which is not a solution of (1) is $(x, y, z) = (4, 31, 11)$ found by Szymiczek [5].

In this paper we give a method for finding positive integer solutions x, y, z of (2), and show two new solutions.

2. Let $1 < x < y, x^2 - 1 = D v^2, y^2 - 1 = D' v'^2$, where D and D' have no square factors. Then we get $D = D'$, because

$$(x^2 - 1)(y^2 - 1) = D D' (v v')^2 = (z^2 - 1)^2.$$

If (u_1, v_1) is the minimal positive solution of the equation

$$u^2 - D v^2 = 1,$$

then all positive solutions are given by (u_k, v_k) for $k = 1, 2, 3, \dots$ where u_k and v_k are the integers defined by

$$u_k + v_k \sqrt{D} = (u_1 + v_1 \sqrt{D})^k. \quad (3)$$

Since we can put $x = u_i, v = v_i, y = u_j, v' = v_j, i < j$, we have

$$D v_i v_j + 1 = z^2. \quad (4)$$

From (3), it follows that

$$u_{k+1} = u_1 u_k + D v_1 v_k,$$

$$v_{k+1} = u_1 v_k + v_1 u_k.$$

Hence, by mathematical induction, $u_k = f_k(u_1)$ is a polynomial of degree k in u_1 , and $v_k/v_1 = g_k(u_1)$ is a polynomial of degree $k-1$ in u_1 . Indeed,

$$g_k(u_1) = \sum_{n=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^n \binom{k-n-1}{n} (2u_1)^{k-2n-1}.$$

Thus, the left side of (4)

$$\begin{aligned} D v_i v_j + 1 &= D v_1 g_i(u_1) \cdot v_1 g_j(u_1) + 1 \\ &= (u_1^2 - 1) g_i(u_1) g_j(u_1) + 1 = F_{ij}(u_1) \end{aligned}$$

is a polynomial of degree $i+j$ in u_1 . If there is a solution (u_1, z) , $u_1 > 1$ of the equation

$$F_{ij}(u_1) = z^2,$$

then $(x, y, z) = (u_i, u_j, z)$ is a solution of (2).

(i) $i=1, j=2$. Since $v_2 = 2u_1 v_1$, we get $g_2(u_1) = 2u_1$, and so $F_{12}(u_1) = (u_1^2 - 1) \cdot 2u_1 + 1 = 2u_1^3 - 2u_1 + 1$. $F_{12}(u_1) = z^2$ has the following three solutions (u_1, z) , $1 < u_1 < 10^5$:

$$(u_1, z) = (3, 7), (4, 11), (155, 2729).$$

Therefore, we obtain the following three solutions of (2), the first of which is a solution of (1), and the last of which is new.

$$(x, y, z) = (3, 17, 7), (4, 31, 11), (155, 48049, 2729).$$

(ii) $i=1, j=4$. Since $v_4 = v_1(8u_1^3 - 4u_1)$, we get

$$F_{14}(u_1) = (u_1^2 - 1)(8u_1^3 - 4u_1) + 1 = 8u_1^5 - 12u_1^3 + 4u_1 + 1.$$

$F_{14}(u_1) = z^2$ has the following solution (u_1, z) , $1 < u_1 < 10^3$:

$$(u_1, z) = (2, 13).$$

Therefore, we obtain the following new solution of (2), where x is even, y and z are odd:

$$(x, y, z) = (2, 97, 13).$$

(iii) In the case of $3 \leq i+j \leq 9$, except above, we get only the following three solutions, all of which are solutions of (1).

i	j		u_1	x	y	z
2	3	$1 < u_1 < 10^3$	3	17	99	41
3	4	$1 < u_1 < 10^2$	3	99	577	239
4	5	$1 < u_1 < 35$	3	577	3363	1393

The equation (2) has no solutions other than above for all integers x and y , where $1 < x < y < 10^3$.

Though we cannot solve (2) completely, we conjecture that there are only three solutions of (2) which are not solutions of (1).

3. We applied our method to the equation

$$(x^2 + 1)(y^2 + 1) = (z^2 + 1)^2,$$

but we could not obtain other solutions than those Williams [6] found.

Our method cannot be applied to the equation

$$(x^2 - e)(y^2 - e) = (z^2 - e)^2$$

where $e \neq \pm 1, \pm 4$. However, our method may be applied to the equation

$$(x^2 - e)(y^2 - e) = \{h(z)\}^2$$

where $e = \pm 1, \pm 4$ and $h(z)$ is a linear or quadratic expression in z .

For example, the equations

$$(x^2 + 1)(y^2 + 1) = (z^2 + t^2)^2,$$

$$(x^2 - 1)(y^2 - 1) = (z^2 - t^2)^2$$

have always integer solutions for all integers t , given by

$$(x, y, z) = (t, 4t^3 + 3t, 2t^2 + 1)$$

$$(x, y, z) = (2t^2 \pm 1, 8t^4 \pm 8t^2 + 1, 4t^3 \pm 3t),$$

respectively.

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REFERENCES

- 1 R. K. Guy: *Unsolved problems in number theory*. New York: Springer-Verlag, 1981, 105.
- 2 J. A. H. Hunter: *Problem 5020*. *Amer. Math. Monthly* 69, 316–317 (1962).
- 3 R. V. Iyer: *Solution 5020*. *Amer. Math. Monthly* 70, 574 (1963).
- 4 A. Schinzel and W. Sierpiński: *Sur l'équation diophantienne $(x^2 - 1)(y^2 - 1) = [((y - x)/2)^2 - 1]^2$* . *El. Math.* 18, 132–133 (1963).
- 5 K. Szymiczek: *On a diophantine equation*. *El. Math.* 22, 37–38 (1967).
- 6 H. C. Williams: *Note on a diophantine equation*. *El. Math.* 25, 123–125 (1970).