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- [5\*] Für Riemann war diese vorwiegend ein anschauliches Hilfsmittel. Erst H. Weyl hat sie in seinem Buch «Die Idee der Riemannschen Fläche» (Teubner, Leipzig 1913) begrifflich scharf gefasst.
- [6\*] Vgl. Dieudonné [8, S. 86], A. Weil [Œuvres II, 532], A. F. Monna [Nieuw Arch. Wisk. (3) 30, 247–57 (1982)], E. P. Hamilton und M. Z. Nashed [J. Funct. Anal. 49, 128–44 (1982)].
- [7\*] Pincherle verdanken wir auch das Wort «Funktionsraum» (er sagt «Funktionalraum» [Encykl. d. math. Wiss. II, 1.2, S. 777], spazio funzionale [Rend. Bologna (2) 1, 85 (1896–7)] espace fonctionnel [Math. Ann. 49, 330 (1897)]. Er sagt in Math. Ann. (ibid.) auch, «Operator» (opérateur) sei 1891 von Carvallo eingeführt worden. «Funktional» (fonctionnelle) als Substantiv stammt von Hadamard, 1904 oder 1905; vgl. [24, S. 251].
- [8\*] Hadamard nennt u. a. C. Bourlet als Vorläufer, der bereits 1897 (Ann. Éc. Norm. Sup. (3) 14, 133–89) ähnliche Ideen publizierte.
- [9\*] Zur Geschichte des Prinzips, das Gauss und Lord Kelvin schon benutzten, siehe [10, 19]. Hilbert hat dann später (1900–01) gezeigt, dass und in welcher Form sich das Prinzip als Beweismethode streng begründen lässt.
- [10\*] Eine bei Neumann verbliebene kleine Lücke hat Lebesgue 1937 (s. Œuvres IV, 151–66) geschlossen.
- [11\*] F. G. Tricomi («Integral Equations», Interscience/Wiley, New York 1957; S. 5) macht dazu eine historisch recht interessante Bemerkung betr. Volterra und Fredholm.

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## Covering the sphere with 11 equal circles

The dual counterpart of the well-known problem of densest spherical circle-packing is the problem of thinnest spherical circle-covering, that is the following: To determine the smallest angular radius  $r_n$  of  $n$  equal circles (or spherical caps) by which the surface of a sphere can be covered without gaps. Contrary to the packing problem, the covering problem has not been intensively investigated. Solutions and conjectures are only known for  $n = 2$  to 10 and 12, 14, 16, 20, 32. References to these results and the literature on the problem of thinnest spherical circle-covering can be found in L. Fejes Tóth's book [3] and in a survey paper by Melnyk, Knop and Smith [4].

The first gap in the sequence of the investigated cases is at  $n = 11$ . The aim of this paper is to fill this gap and to present a good construction for covering the sphere with 11 equal circles.

To a covering system of the circles a graph is associated as suggested by L. Fejes Tóth [3]. The graph is a bipartite graph. It contains two kinds of vertices. The vertices of the first kind are the centres of the spherical circles and the vertices of the second kind are the points of the perimeters of the circles in which the spherical point is only just covered. (In the figures, the vertices of the first kind will be marked by small circles but the vertices of the second kind will not have any special mark.)

The edges of the graph are the shorter great circle arcs joining the centres and the just covering perimetric points of the circles. As a consequence of equality of the circles, all the edges of the graph are of equal length.

For  $n = 10$  the solution is due to G. Fejes Tóth [1]. The centres of the circles form a regular bipyramided square antiprism and the angular radius of the circles is  $r_{10} = 42^\circ 18' 28.2''$ . The graph of the arrangement of the circles can be seen in a simplified stereographic projection in figure 1 where  $A$  and  $J$  denote the vertices of the pyramids.

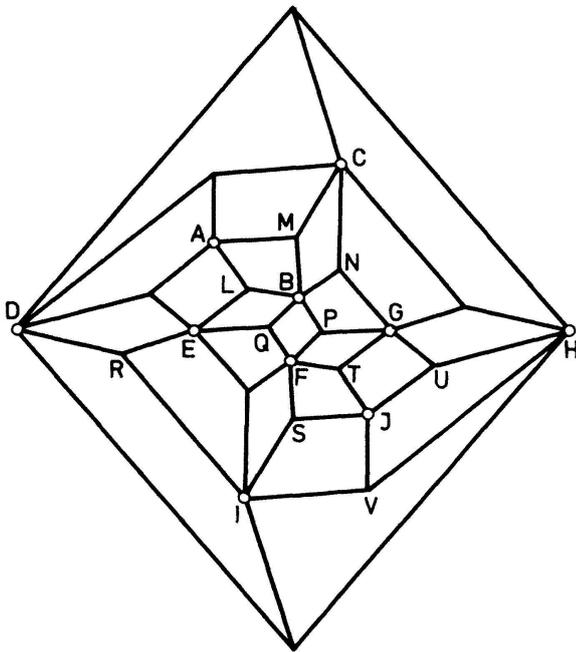


Figure 1. The graph of the best covering with 10 circles.

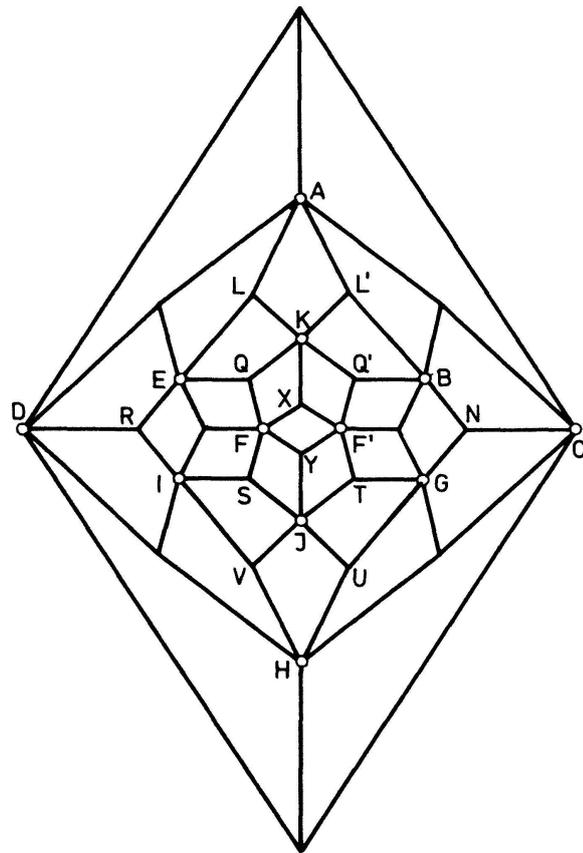


Figure 2. The graph of the best covering with 12 circles.

For  $n = 12$  the solution is due to L. Fejes Tóth [2]. The centres of the circles form a regular icosahedron and the angular radius of the circles is  $r_{12} = 37^\circ 22' 38.5''$ . The graph of the arrangement of the circles can be seen in a simplified stereographic projection in figure 2. One can see that by removing the vertices  $K, X, Y$  together with the edges joining in these vertices from the graph of the best covering with 12 circles in figure 2 and by shifting the vertices  $F, L, Q$  and  $F', L', Q'$ , respectively, into coincidence with simultaneous increase in the edge-lengths we obtain the graph of the best covering with 10 circles in figure 1.

Recently it was shown in [5] that by removing the second-kind vertices  $L, L', N, Q, Q', R, S, T, U, V, X, Y$  of figure 2 together with the edges joining in these vertices and by shifting the first-kind vertices  $F$  and  $F'$  into coincidence with a simultaneous increase in the edge-lengths we obtain the arrangement in figure 3 where the minimum angular radius of the circles is  $r_{11} = 41^\circ 29' 28.0''$ .

It should be noted that the same result is obtained if the second-kind vertices  $L, N, Q, R, S, T, U, V$  together with the edges joining in these vertices are removed from the graph in figure 1, and the modified graph is supplemented by an additional first-kind vertex  $K$ , then the edge-lengths are simultaneously decreased.

A better configuration can be arrived at in the following way. Consider the graph of the covering by 10 circles in figure 1. Remove the second-kind vertices  $M$  and  $S$  together with the edges joining in these vertices and the edges  $BP$  and  $FQ$ . Supplement the graph by an additional first-kind vertex  $K$  and additional edges  $QK$  and  $PK$ . The obtained graph is movable with finite motion. Move the graph into a position in which, in stereographic

projection, the graph is centrally symmetric with respect to point  $K$ . In this position the length of the edges of the graph, that is, the angular radius of the circles is a function of three independent angles  $\alpha, \beta, \gamma$  of the graph. Thus, the task is to minimize a trigonometric function of three variables. The numerical computation was executed by means of the gradient method. The minimum of the angular radius of the circles was obtained at the angles  $\alpha = 87^\circ 12' 01.2''$ ,  $\beta = 41^\circ 16' 12.8''$ ,  $\gamma = 102^\circ 43' 56.0''$  and its value is

$$r_{11} = 41^\circ 25' 37.9'' .$$

The graph of this new arrangement can be seen in a simplified stereographic projection in figure 4 and the angles  $\alpha, \beta, \gamma$  are also shown there.

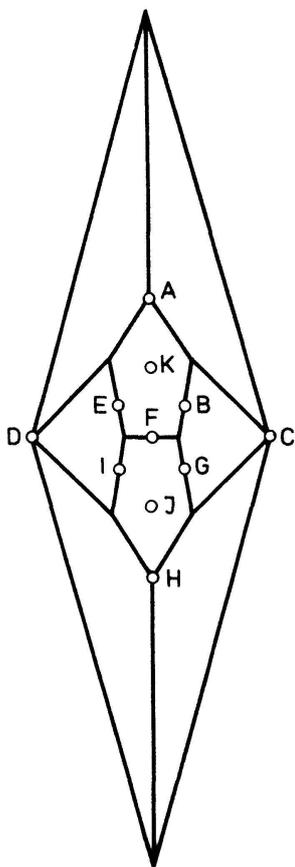


Figure 3. The graph of the covering with 11 circles.

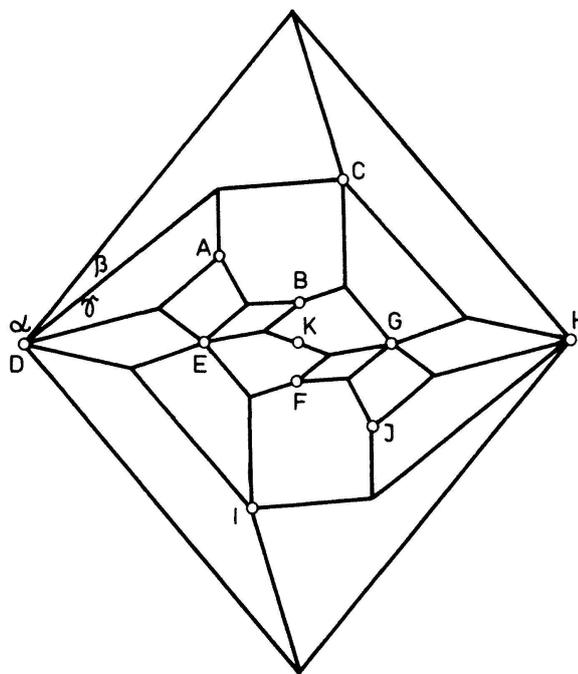


Figure 4. The graph of the improved covering with 11 circles.

Comparing the two graphs in figures 3 and 4 resulting in local minima, we can observe that the better arrangement is of lower symmetry. This is in agreement with a remark made in [4] that with increasing  $n$  'there will be an increasing number of local extrema and an increasing probability that the global minimum will be of low symmetry'.

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## Kleine Mitteilungen

### Die Einfachheit der Gruppe $A_5$

Der folgende Beweis der Einfachheit der Gruppe  $A_5$  benötigt ausser dem Satz von Lagrange nur simpelste Kombinatorik und folgendes unmittelbare Korollar aus dem Isomorphiesatz:

$$(*) \quad G \text{ Gruppe, } N \trianglelefteq G, \quad x \in G, \quad (o(x), |G : N|) = 1 \Rightarrow x \in N.$$

$A_n$  sei als Kern des Signumshomomorphismus eingeführt (eine elegante Darstellung dazu findet sich in [1, 2.1.3]). Dann enthält  $A_5$  alle Doppeltranspositionen, wegen  $|S_5 : A_5| = 2$  aber nach (\*) auch alle Elemente ungerader Ordnung von  $S_5$ , also mindestens

$$\begin{aligned} 20 &= 2 \cdot \binom{5}{3} \text{ Permutationen der Form } (ijk) \text{ (Ordnung 3),} \\ 24 &= 4! \text{ Permutationen der Form } (ijklm) \text{ (Ordnung 5),} \\ 15 &= 5 \cdot \binom{3}{2} \text{ Permutationen der Form } (ij)(kl) \text{ (Ordnung 2).} \end{aligned}$$

Wegen  $|A_5| = 60$  schöpfen diese mit dem Einselement ganz  $A_5$  aus.

Sei nun  $1 < N \triangleleft A_5$  und  $a := |A_5 : N| \neq 1$ . Es gilt:  $|N| \mid 60$ . Falls  $3 \nmid a$ , so folgt aus (\*):  $21 \leq |N|$ , also  $|N| = 30$ , damit  $5 \nmid a$  und wiederum mit (\*) der Widerspruch  $30 = |N| \geq 45$ . Also gilt  $3 \mid a$ , und genauso  $5 \mid a$ . Daher folgt  $|N| \mid 4$ . Die Möglichkeit  $|N| = 4$  führt, wieder nach (\*), zu dem Widerspruch  $4 = |N| \geq 16$ . Also ist  $|N| = 2$ , d. h.  $N = \{id, (ij)(kl)\}$  aufgrund der Vollständigkeit der obigen Elementliste. Aber dann ist

$$N(ijk) = \{(ijk), (ikl)\} \neq \{(ijk), (jlk)\} = (ijk)N,$$

endgültig mit Widerspruch.

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