

**Zeitschrift:** Elemente der Mathematik  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 40 (1985)  
**Heft:** 4

**Artikel:** A point set everywhere dense in the plane  
**Autor:** [s.n.]  
**DOI:** <https://doi.org/10.5169/seals-38834>

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# ELEMENTE DER MATHEMATIK

Revue de mathématiques élémentaires – Rivista di matematica elementare

*Zeitschrift zur Pflege der Mathematik  
und zur Förderung des mathematisch-physikalischen Unterrichts*

El. Math.

Band 40

Nr. 4

Seiten 81–104

Basel, 10. Juli 1985

## A point set everywhere dense in the plane

**Dedicated to Professor L. Fejes-Tóth on his seventieth birthday**

One day the famous Hungarian geometer László Fejes-Tóth could not find his compass. He searched through every nook and cranny of the house without success and, for it was a nice old piece given by his wife as a present many years ago, he felt very sad over the loss. However, a few days later his wife who was obviously better at searching found the compass and so she managed to present her husband with the same gift for the second time. (This is a very special feminine virtue.)

Professor Fejes-Tóth who has always been passionately interested in all sorts of properties of circle arrangements and did a lot of important work in this field, was quite delighted. He took the compass eagerly in his hand, opened its arms and started absent-mindedly drawing equal circles on a sheet of paper on his desk. He had to stop this action before long, because after a while no single spot of the paper had been left blank. Being a dedicated mathematician he immediately decided that this could not be just an accident. He scrutinized the picture and then he suddenly realized that he had been following a very simple and natural strategy of drawing:

- (i) Take two points  $O_1, O_2$  having mutual distance at most 2, and set  $S_1 = \{O_1, O_2\}$ .
- (ii) If  $S_i$  ( $i \geq 1$ ) has already been determined then add to the picture all unit circles whose centres belong to  $S_i$  and which have not been drawn in before. Further, let  $S_{i+1}$  be defined as the union of  $S_i$  and the set of all intersection points of units circles whose centres are in  $S_i$ .

If  $|O_1 O_2|$  (i.e. the distance of  $O_1$  and  $O_2$ ) is 1,  $\sqrt{3}$  or 2 then, having taken a few steps, we obtain the following picture (see Fig. 1):

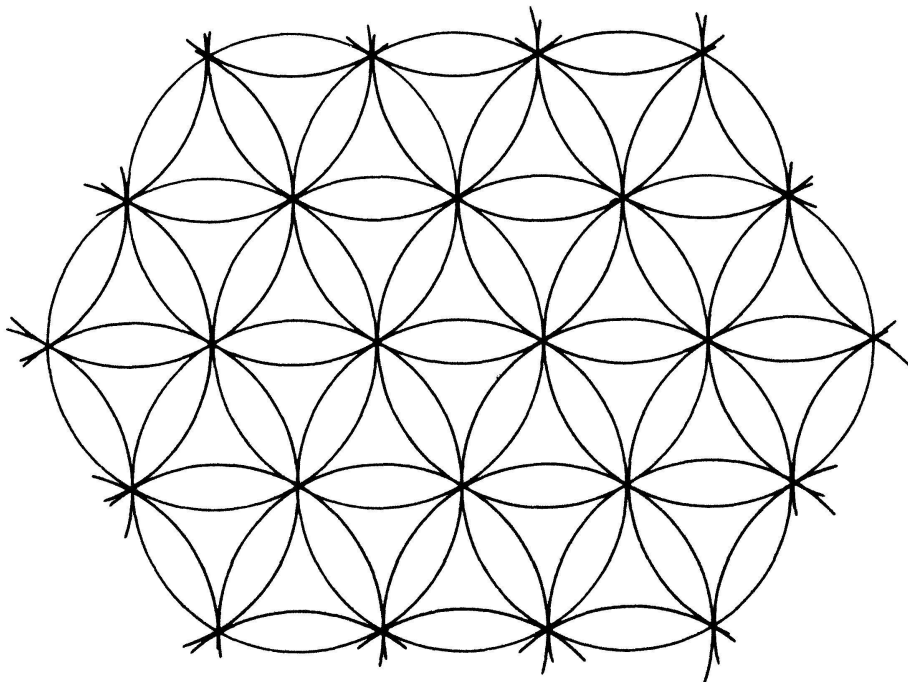


Figure 1

That is, in these cases  $\bigcup_{i=1}^{\infty} S_i$  consists of all vertices of a *regular triangle-lattice* of side length 1.

Professor Fejes-Tóth who was not lucky enough to choose one of these three numbers for  $|O_1 O_2|$  got a somewhat uglier picture. In particular, his sets  $S_i$  filled up the space more and more 'densely' ( $i \rightarrow \infty$ ).

**Definition.** An infinite point set  $S$  is said to be everywhere dense in the plane if every circular disc contains at least one element of  $S$ .

After a little meditation, L. Fejes-Tóth came up with the following conjecture:

**Theorem 1.** Let the sets  $S_i$  ( $i \geq 1$ ) be defined as above. Then,  $\bigcup_{i=1}^{\infty} S_i$  is either identical with the set of vertices of a regular triangle-lattice of side length 1, or it is everywhere dense in the plane.

Our proof is based on the elementary

**Lemma 1.** Let  $ABC$  be an equilateral triangle with side length at most 2, and suppose  $A, B \in S_i$  for some  $i \geq 1$ . Then, we have  $C \in S_{i+3}$ .

**Proof:** It is obviously sufficient to prove the assertion for the case  $|AB| < 2$ .

Let us define four additional points (see Fig. 2) by

$O$ :  $|OA| = |OB| = 1$  and the line  $AB$  separates  $O$  and  $C$ ;

$A'$ :  $|A'A| = |A'O| = 1$  and the line  $AO$  does not separate  $A'$  and  $C$ ;

$B'$ :  $|B'B| = |B'O| = 1$  and the line  $BO$  does not separate  $B'$  and  $C$ ;

$B''$ :  $|B''B| = |B''O| = 1$  and the line  $BO$  separates  $B''$  and  $C$ .

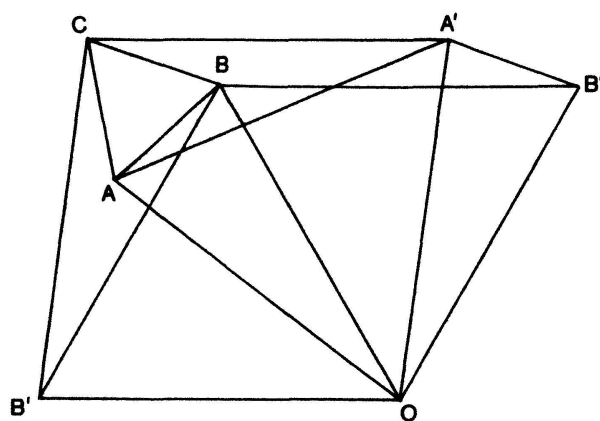


Figure 2

Observe now that  $A'B''$  is the image of  $AB$  at a rotation about  $O$  with angle  $\pi/3$ . Consequently,  $A'B''BC$  is a parallelogram. Then we have  $|CA'| = |BB''| = 1$  and, similarly,  $|CB'| = 1$ .

Thus,  $C$  is an intersection point of the unit circles centred at  $A'$  and  $B'$ , and using the fact that  $O \in S_{i+1}$  and  $A', B' \in S_{i+2}$  (by the definitions) we get  $C \in S_{i+3}$ .

Proof of theorem 1: First, we note that lemma 1 has the immediate consequence that, if  $A$  and  $B$  are two elements of  $S = \bigcup_{i=1}^{\infty} S_i$  with  $|AB| \leq 2$ , then  $S$  contains all vertices of the regular triangle-lattice determined by  $A$  and  $B$ . In view of this, we can clearly assume

$$\inf\{|AB| : A, B \in S\} = \delta > 0,$$

otherwise  $S$  is everywhere dense.

Let us now choose two points  $A, B \in S$  such that  $|AB| < (3/2)\delta$ . Then  $S$  must be the set of vertices of the regular triangle-lattice determined by  $A$  and  $B$ ; for if there were any additional point  $P \in S$  in a triangular cell  $A'B'C'$  of the lattice, then at least one of the distances  $|PA'|$ ,  $|PB'|$ ,  $|PC'|$  would be shorter than  $|A'B'|/\sqrt{3} = |AB|/\sqrt{3} < \delta$ , a contradiction. This yields e.g. that  $|AB| = \delta \leq 1$ .

It remains to prove that  $\delta = 1$ . Suppose to the contrary that  $\delta < 1$ , and let  $A^*$  denote the reflection of  $A$  about  $B$ . Further, let  $O$  and  $O^*$  be two points satisfying  $O: |OA| = |OB| = 1$ ,  $O^*: |O^*A| = |O^*A^*| = 1$  and the line  $AB$  does not separate  $O$  and  $O^*$ . It is easy to check now that  $|OO^*| < \delta$ , the desired contradiction.

Having seen our proof of theorem 1, L. Fejes-Tóth immediately asked whether or not a similar assertion is valid for arbitrary centrally symmetric closed convex curves, instead of circles. In what follows, we will settle this question in the affirmative.

Given any centrally symmetric strictly convex closed curve  $K$  and any point  $P$  in the plane, let  $K(P)$  denote the translate of  $K$  with centre  $P$ .

(i') Let  $O_1$  and  $O_2$  be two points such that  $K(O_1)$  and  $K(O_2)$  have at least one point in common, and set  $S_1 = \{O_1, O_2\}$ .

(i'') If  $S_i$  ( $i \geq 1$ ) has already been determined then let  $S_{i+1}$  be defined as the set of all points which are either in  $S_i$  or in  $K(P_1) \cap K(P_2)$  for some  $P_1, P_2 \in S_i$ .

If  $O_2 \in K(O_1)$  then, following this procedure, we will find again that  $\bigcup_{i=1}^{\infty} S_i$  is a *triangle-lattice* whose cells are congruent to the triangle  $O_1O_2O_3$ , where  $O_3$  denotes one of the intersection points of  $K(O_1)$  and  $K(O_2)$ . Any lattice obtained in this way is said to be *K-regular* (see Fig. 3).

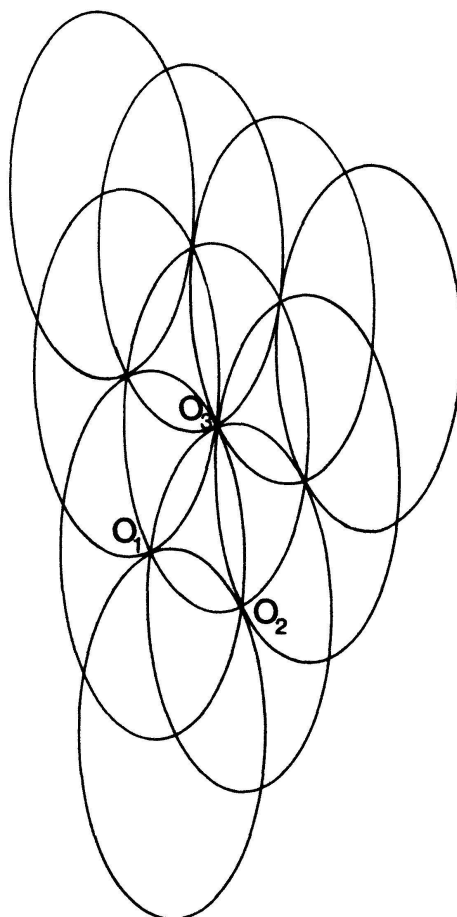


Figure 3

The following generalization of theorem 1 is true:

**Theorem 2.** *Let  $K$  be a centrally symmetric strictly convex closed curve in the plane, and let the sets  $S_i$  ( $i \geq 1$ ) be defined by (i') and (i''). Then  $S = \bigcup_{i=1}^{\infty} S_i$  is either identical with the vertex set of a  $K$ -regular lattice, or it is everywhere dense in the plane.*

The core of the proof is contained in

**Lemma 2.** *Let  $A, B, C$  be three elements of  $S$ . Then all vertices of the triangle-lattice determined by these points belong to  $S$ .*

To see this we need three simple but useful observations.

**Proposition 1.** *Let  $P, Q \in S$  such that  $P \in K(Q)$ . Then  $Q + \overrightarrow{PQ}$ , i.e. the reflection of  $P$  about  $Q$ , is an element of  $S$ .*

Proof: Let  $R_1 \in K(P) \cap K(Q)$ , and let  $R_2$  denote the intersection point of  $K(R_1)$  and  $K(Q)$  distinct from  $P$ . Further, let  $R_3$  be the intersection point of  $K(R_2)$  and  $K(Q)$  different from  $R_1$ . We obviously have  $R_3 = Q + \overrightarrow{PQ} \in S$ .

**Proposition 2.** *Let  $O_1$  and  $O_2$  denote, as above, the starting points of our algorithm (see (i')). Then  $O_2 + \overrightarrow{O_1O_2} \in S$ .*

Proof: Let  $K(O_1) \cap K(O_2) = \{P_1, P_2\}$ . By proposition 1, we have  $P_1 + \overrightarrow{O_1P_1}, O_2 + \overrightarrow{P_2O_2} \in S$ , hence, applying proposition 1 to these two latter points, we obtain  $(O_2 + \overrightarrow{P_2O_2}) + \overrightarrow{O_1P_1} = O_2 + \overrightarrow{O_1O_2} \in S$ .

**Proposition 3.** *Let  $O_1, O_2$  be the starting points of our algorithm, and suppose  $P, Q \in S$ . Then  $P \pm \overrightarrow{O_1Q}$  (i.e. the translates of  $P$  by the vectors  $\overrightarrow{O_1Q}$  and  $\overrightarrow{QO_1}$ , resp.) are also elements of  $S$ .*

Proof: By proposition 2, we have  $O_2 + \overrightarrow{O_1O_2} \in P$ . Thus, starting our algorithm with the points  $O'_1 = O'_2, O'_2 = O_2 + \overrightarrow{O_1O_2}$  (instead of  $O_1$  and  $O_2$ ), we obtain a set  $S' \subseteq S$ . Taking into account that  $P \in S$ , we get  $P + \overrightarrow{O_1O_2} \in S'$ .

Let us start next with  $O''_1 = P, O''_2 = P + \overrightarrow{O_1O_2}$ . Using the fact that  $Q \in S$ , we obtain now  $Q \pm \overrightarrow{O_1P} = P + \overrightarrow{O_1Q} \in S'' \subseteq S$ . The relation  $P - \overrightarrow{O_1Q} \in S$  can be established similarly.

Proof of lemma 2: It is sufficient to show e.g. that  $A + \overrightarrow{BC} \in S$ . But this is an immediate consequence of proposition 3 and the fact that  $A + \overrightarrow{BC}$  can be written in the form  $A + \overrightarrow{O_1C} - \overrightarrow{O_1B}$ .

The proof of theorem 2 (based on lemma 2) is now a relatively easy exercise which can be left to the reader.

For those familiar with the notion of Minkowski spaces, we remark that the convex curve  $K$  in theorem 2 is, in fact, a *unit circle* in the corresponding Minkowski metrics. Using this terminology, steps (i') and (i'') of our 'drawing algorithm' reduce to (i) and (ii), resp.

Finally, note that similar assertions can be proved for circles in the hyperbolic plane and on the sphere.

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