Coordinate free definitions of cross ratio

1. Introduction

Let P_1, P_2, P_3, P_4 be four distinct points on the projective line $\mathbf{P}(V)$, i.e., four distinct one-dimensional linear subspaces of the two-dimensional vector space V over a field K. Their cross ratio $[P_1, P_2; P_3, P_4]$ usually is defined by means of coordinates (see, e.g., Berger [1], p. 153–155). It might be of interest of show that an elegant coordinate free alternative is possible. Our main tools are parallel projections in V on lines through the origin. We shall define cross ratio as the trace of the composition of two such endomorphisms. As a matter of fact, it is well-known that the notion of trace is a coordinate free concept.

The derivation of a few simple results on traces in section 3 paves the way for a complete treatment of cross ratio in section 4. Next we show that our definition is equivalent to the usual ones with coordinates, and finally we present two alternative possibilities for invariant definitions of cross ratio.

1.1. We shall employ the following elementary properties of the trace $\mathrm{tr}(\varphi)$, where φ is an endomorphism of V:

(i) $\mathrm{tr}(\varphi + \psi) = \mathrm{tr}(\varphi) + \mathrm{tr}(\psi)$,

(ii) $\mathrm{tr}(\varphi \cdot \psi) = \mathrm{tr}(\psi \cdot \varphi)$,

(iii) if φ has two eigenvalues λ_1, λ_2, then $\mathrm{tr}(\varphi) = \lambda_1 + \lambda_2$ (recall that V is two-dimensional).

2. The projections π_j^i

2.1 Definition

For $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, let $\pi_j^i \colon V \to V$ be the projection on P_j in the direction P_i.
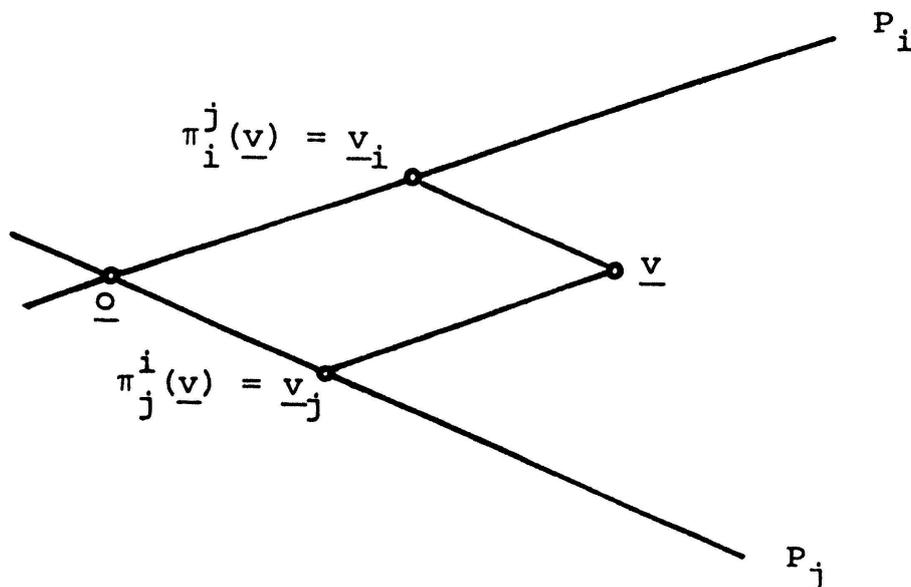

Figure 1

(Recall that any $\underline{v} \in V$ can be written in a unique way as $\underline{v} = \underline{v}_i + \underline{v}_j$ with $\underline{v}_i \in P_i, \underline{v}_j \in P_j$. Then, by definition, $\pi_j^i(\underline{v}) = \underline{v}_j$.)

2.2. We list some obvious properties of the π_j^i:

(i) $(\pi_j^i)^2 = \pi_j^i$,

(ii) $\pi_j^i + \pi_i^j = id$ (the identity mapping),

(iii) $\operatorname{tr}(\pi_j^i) = 1$ (cf. 1.1 (iii)).

3. The endomorphisms $\pi_j^i \pi_l^k$

3.1. The eigenspaces of the endomorphisms $\pi_4^1 \pi_3^2$ obviously are P_2 and P_4; the corresponding eigenvalues being 0 and λ for some $\lambda \neq 0$. Thus $\lambda = \operatorname{tr}(\pi_4^1 \pi_3^2)$. Figure 2 illustrates the effect of $\pi_4^1 \pi_3^2$ on an eigenvector $\underline{p} \in P_4$.

$$\lambda \underline{p} = \pi_4^1 \pi_3^2 (\underline{p})$$

P_3

$\pi_3^2 (\underline{p})$

P_2

$\underline{p}$

P_1

P_4
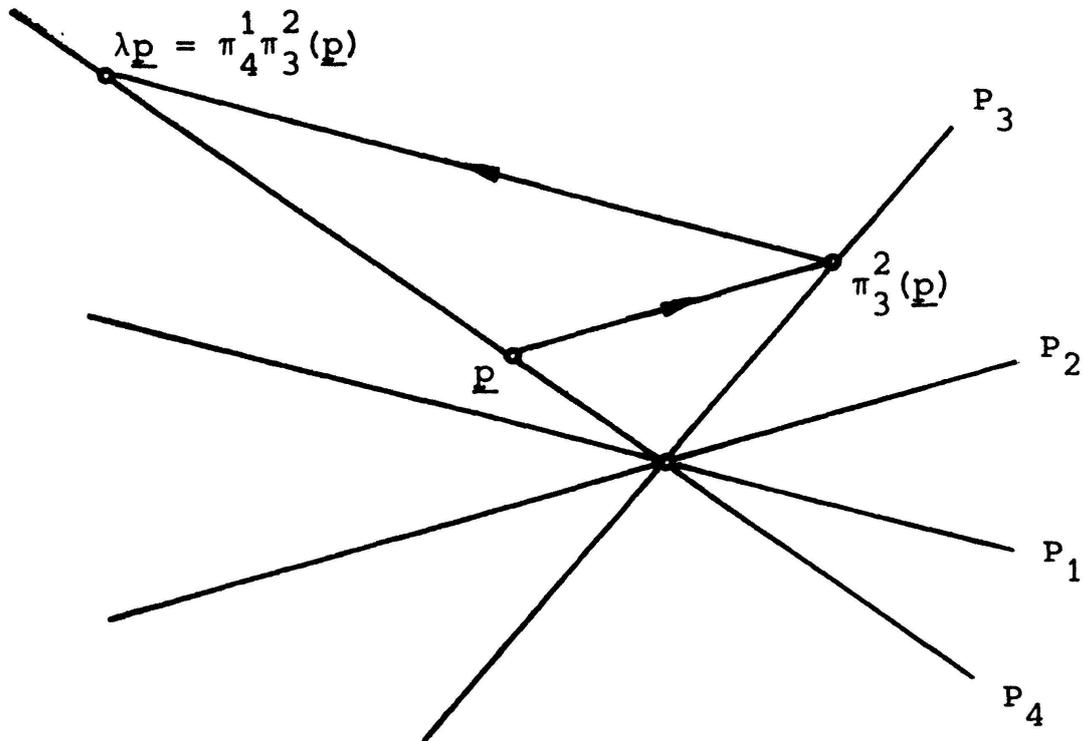
Figure 2

3.2 *Lemma*

$$\operatorname{tr}(\pi_4^1 \pi_3^2) = \operatorname{tr}(\pi_3^2 \pi_4^1), \tag{i}$$

$$\operatorname{tr}(\pi_4^1 \pi_3^2) \cdot \operatorname{tr}(\pi_4^2 \pi_3^1) = 1, \tag{ii}$$

$$\operatorname{tr}(\pi_4^1 \pi_3^2) + \operatorname{tr}(\pi_4^1 \pi_2^3) = 1. \tag{iii}$$

Proof:

(i) This instance of 1.1 (ii) can also be verified directly:

$$\pi_3^2 \pi_4^1 (\pi_3^2(\underline{p})) = \pi_3^2(\pi_4^1 \pi_3^2(\underline{p})) = \pi_3^2(\lambda \underline{p}) = \lambda \pi_3^2(\underline{p})$$

($\underline{p} \in P_4$, see fig. 2), so the nonzero eigenvalue of $\pi_3^2 \pi_4^1$ is λ.

(ii) $\pi_4^2 \pi_3^1$ maps $\pi_4^1 \pi_3^2(\underline{p})$ onto $\underline{p}$ (reverse the arrows in fig. 2!). So the nonzero eigenvalue

of $\pi_4^2 \pi_3^1$ is $\dfrac{1}{\lambda}$.

(iii) $\pi_4^1 \pi_3^2 + \pi_4^1 \pi_2^3 = \pi_4^1 (\pi_3^2 + \pi_2^3) = \pi_4^1$.

Now apply 1.1 (i) and 2.2 (iii).

3.3 Corollary

Since the permutations (12) (34), (12) and (23) generate the group of permutations of $\{1,2,3,4\}$, we can express $\mathrm{tr}\,(\pi_j^i \pi_l^k)$ as a function of $\mathrm{tr}\,(\pi_4^1 \pi_3^2)$ for any four distinct indices i, j, k, l. In particular, we have

$$\mathrm{tr}\,(\pi_4^1 \pi_3^2) = \mathrm{tr}\,(\pi_3^2 \pi_4^1) = \mathrm{tr}\,(\pi_1^4 \pi_2^3) = \mathrm{tr}\,(\pi_2^3 \pi_1^4).$$

4. Cross ratio

4.1 Definition

The *cross ratio* $[P_1, P_2; P_3, P_4]$ of the ordered quadruple (P_1, P_2, P_3, P_4) is the element

$$[P_1, P_2; P_3, P_4] = \mathrm{tr}\,(\pi_4^1 \pi_3^2) \in K.$$

4.2 Proposition

$$[P_1, P_2; P_3, P_4] = [P_2, P_1; P_4, P_3] = [P_4, P_3; P_2, P_1] = [P_3, P_4; P_1, P_2], \tag{i}$$
$$[P_1, P_2; P_3, P_4] \cdot [P_2, P_1; P_3, P_4] = 1, \tag{ii}$$
$$[P_1, P_2; P_3, P_4] + [P_1, P_3; P_2, P_4] = 1. \tag{iii}$$

Proof: 3.2, 3.3.

4.3 Proposition

If $\psi : V \to V'$ is an isomorphism of V onto another K-vector space V', and if $P_i' = \psi\,(P_i)$ $(i = 1,2,3,4)$, then

$$[P_1, P_2; P_3, P_4] = [P_1', P_2'; P_3', P_4'].$$

Proof: $\bar{\pi}_j^i = \psi \pi_j^i \psi^{-1}$ is the projection on P_j' along P_i', so

$$[P_1', P_2'; P_3', P_4'] = \mathrm{tr}\,(\bar{\pi}_4^1 \bar{\pi}_3^2) = \mathrm{tr}\,(\psi \pi_4^1 \pi_3^2 \psi^{-1}) = \mathrm{tr}\,(\psi^{-1} \psi \pi_4^1 \pi_3^2)$$
$$= \mathrm{tr}\,(\pi_4^1 \pi_3^2) = [P_1, P_2; P_3, P_4].$$

4.4 Remark

From $P_1 \neq P_2$ and $P_3 \neq P_4$ it follows that $\lambda = [P_1, P_2; P_3, P_4] \neq 1$ (cf. fig. 2), and from $P_1 \neq P_3$, $P_2 \neq P_4$ it follows that $\lambda = [P_1, P_2; P_3, P_4] \neq 0$.

Figure 2 strongly suggests how to *define* $[P_1, P_2; P_3, P_4]$ if two of the P_i coincide. Indeed, one usually defines

$$[P_1, P_1; P_3, P_4] = [P_1, P_2; P_3, P_3] = 1,$$
$$[P_1, P_2; P_1, P_4] = [P_1, P_2; P_3, P_2] = 0$$

and also

$$[P_1, P_2; P_3, P_1] = [P_1, P_2; P_2, P_4] = \infty.$$

4.5 Proposition

If $[P_1, P_2; P_3, P_4] = [Q_1, Q_2; Q_3, Q_4]$ and if $P_i = Q_i$ for three values of i, then $P_i = Q_i$ for all four values of i.

Proof: On account of proposition 4.2 it is sufficient to prove that from $[P_1, P_2; P_3, P_4]$ $= [P_1, Q_2; P_3, P_4]$ it follows that $P_2 = Q_2$. Renaming, for convenience, $Q_2 = P_5$, we have $\mathrm{tr}(\pi_4^1 \pi_3^2) = \mathrm{tr}(\pi_4^1 \pi_3^5)$.
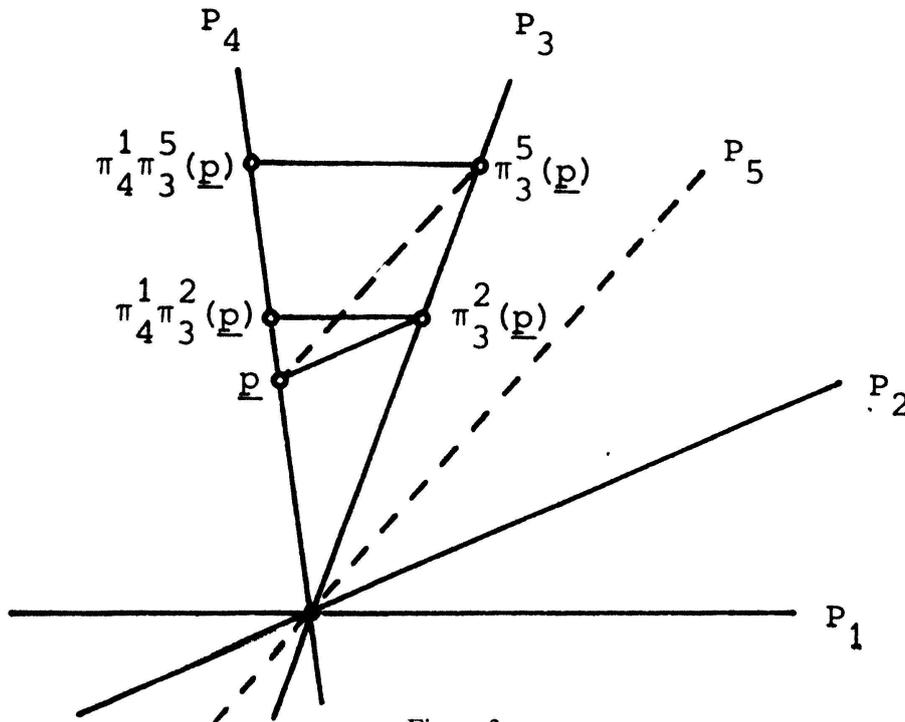

Figure 3

This implies that $\pi_4^1 \pi_3^2(\underline{p}) = \pi_4^1 \pi_3^5(\underline{p})$ for all eigenvectors $\underline{p} \in P_4$, whence $P_2 = P_5 (= Q_2)$.

4.6 Corollary

If $P_1, P_2, P_3, P_4 \in \mathbf{P}(V)$, $Q_1, Q_2, Q_3, Q_4 \in \mathbf{P}(V')$, where V' is another two-dimensional K-vector space, and if

$$[P_1, P_2; P_3, P_4] = [Q_1, Q_2; Q_3, Q_4]$$

then there exists an isomorphism $\psi: V \to V'$ such that $\psi(P_i) = Q_i$ $(i = 1, 2, 3, 4)$.

Proof: There exists an isomorphism $\psi: V \to V'$ such that

$$\psi(P_i) = Q_i \quad (i = 1, 2, 3).$$

By proposition 4.3 we have

$$([Q_1, Q_2; Q_3, Q_4] =)[P_1, P_2; P_3, P_4]$$
$$= [\psi(P_1), \psi(P_2); \psi(P_3), \psi(P_4)] = [Q_1, Q_2; Q_3, \psi(P_4)]$$

whence $\psi(P_4) = Q_4$.

5. Calculation of cross ratio in coordinates

5.1 *Proposition*

If, with respect to some coordinate system in V, the one-dimensional subspaces P_i are spanned by vectors with coordinates (x_i, y_i), then

$$[P_1, P_2; P_3, P_4] = \frac{\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} \begin{vmatrix} x_2 & x_4 \\ y_2 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & x_4 \\ y_1 & y_4 \end{vmatrix} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}}.$$

Proof: Writing $[ij] = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$ we have

$$\pi_j^i = \frac{1}{[ij]} \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_j & -x_j \\ -y_i & x_i \end{pmatrix} = \frac{1}{[ij]} \begin{pmatrix} -x_j y_i & x_j x_i \\ -y_j y_i & y_j x_i \end{pmatrix}$$

so

$$\text{tr}(\pi_4^1 \pi_3^2) = \frac{1}{[14][23]} (x_4 y_1 x_3 y_2 - x_4 x_1 y_3 y_2 - y_4 y_1 x_3 x_2 + y_4 x_1 y_3 x_2) = \frac{[13][24]}{[14][23]}.$$

5.2. If, in particular, one chooses coordinates such that $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (0, 1)$, $(x_3, y_3) = (1, 1)$, $(x_4, y_4) = (\lambda, 1)$ for some $\lambda \in K$ (this is always possible), then we have

$$[P_1, P_2; P_3, P_4] = \lambda.$$

This also shows that for any $\lambda \in K$, $\lambda \neq 0, 1$, there are four distinct P_1, P_2, P_3, P_4 such that $[P_1, P_2; P_3, P_4] = \lambda$.

6. Alternative

The following formulae provide alternative invariant definitions of the cross ratio $\lambda = [P_1, P_2; P_3, P_4]$:

$$\pi_4^1 \pi_3^2 + \pi_2^3 \pi_1^4 = \lambda \cdot id, \tag{i}$$
$$\det(id - \pi_4^1 \pi_2^3) = \lambda. \tag{ii}$$

Proof:

(i) P_2 and P_4 are eigenspaces of $\pi_4^1 \pi_3^2 + \pi_2^3 \pi_1^4$ with corresponding eigenvalues $0 + \lambda$ and $\lambda + 0$. It follows that V as a whole is an eigenspace with eigenvalue λ.

(ii) The eigenspaces of $id - \pi_4^1 \pi_2^3$ are P_3 and P_4 with corresponding eigenvalues 1 and $1 - (1 - \lambda) = \lambda$ (cf. lemma 3.2 (iii)).

Juriaan Simonis and Jan van de Craats

REFERENCE

1 M. Berger: Géométrie 1. Cedic Fernand Nathan, Paris 1977.

Zur Kantenwinkelsumme der regulären Pyramiden

1. Einleitung

Eine Pyramide heisst *regulär,* wenn ihre Basis ein regelmässiges Vieleck ist und die Spitze auf dem Mittellot der Basis liegt. Sie ist bestimmt durch die Anzahl n der Ecken der Basis ($n \in \mathbf{N}, n > 2$), den Radius r des Umkreises der Basis ($r \in \mathbf{R}, r > 0$) und die Höhe h der Pyramide ($h \in \mathbf{R}, h > 0$). Der Mantel besteht aus n Dreiecken, die gegen die Basis alle unter dem gleichen Winkel φ geneigt sind; je zwei benachbarte dieser Dreiecke sind ebenfalls unter immer dem gleichen Winkel ψ gegeneinander geneigt. Für die Summe Σ der Kantenwinkel gilt

$$\Sigma = n(\varphi + \psi). \tag{1}$$

Da Winkel unter Ähnlichkeitstransformationen invariant bleiben, kann man die Untersuchung von Σ auf den Fall

$$r = 1 \tag{2}$$

beschränken. Bei festem n gilt dann offensichtlich, dass in Abhängigkeit von h

(a) $\varphi = \varphi_n(h)$ monoton von 0 bis $\frac{1}{2}\pi$ wächst und
(b) $\psi = \psi_n(h)$ monoton von π bis

$$\alpha_n = \left(1 - \frac{2}{n}\right) \cdot \pi. \tag{3}$$

fällt.