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Kleine Mitteilungen

On the Limits of Simple Means

1. Let $\{a_n\}$ $(n=1,2,\ldots)$ be an infinite sequence of positive real numbers. If $a_n \to A$ as $n \to \infty$, then both the arithmetic mean

$$A_n(a) = (a_1 + a_2 + \ldots + a_n)/n$$

and the geometric mean

$$G_n(a) = \sqrt[n]{a_1 a_2 \ldots a_n}$$

converge to A. But when a_n is not convergent, it does not necessarily follow that $G_n(a) \to A$ from $A_n(a) \to A < \infty$. This is easily seen, for instance, from the example $a_n = 2 + (-1)^n$. So we may naturally ask when it really does. An obvious sufficient condition to ensure that $G_n(a) \to A$ when $A_n(a) \to A$ will be $H_n(a) \to A$, where $H_n(a)$ is the harmonic mean

$$H_n(a) = n \left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right).$$

That this is not necessary at all can be seen e.g. from

$$a_n = 1 \; (n \neq 2^\nu), \qquad \frac{1}{n} \; (n = 2^\nu), \tag{1}$$

in which case $A_n(a) \to 1$ and $G_n(a) \to 1$ but $H_n(a) \to \frac{1}{3}$.
It is to be remarked that $\lim\limits_{n\to\infty} A_n(a) = \lim\limits_{n\to\infty} G_n(a) = \lim\limits_{n\to\infty} H_n(a) = A < \infty$ does not imply $a_n \to A$. Consider e.g. $a_n = 1$ if $n \neq 2^\nu$, $= 2$ if $n = 2^\nu$. However, the following theorem holds.

Theorem 1. *Let* $\{a_n\}$ $(n=1,2,\ldots)$ *be an infinite sequence of positive real numbers, and suppose that*

$$\lim_{n\to\infty} A_n(a) = A < \infty, \tag{2}$$

$$\liminf_{n\to\infty} a_n > 0. \tag{3}$$

Then

$$\lim_{n\to\infty} G_n(a) = A \tag{4}$$

if and only if for all positive constants p, q

$$\lim_{n\to\infty} H_n(pa+q) = pA + q. \tag{5}$$

The assertion is invalid if we drop the extra condition (3), *or admit* $q=0$ *or* $A = \infty$.

We notice that $H_n(a) \to A < \infty$ does not always imply $H_n(pa+q) \to pA+q$. In fact we can obtain the theorem below from which Theorem 1 can easily be deduced.

Theorem 2. *Let* $\{a_n\}$ $(n = 1, 2, \ldots)$ *be a bounded sequence of positive real numbers such that*

$$\lim_{n\to\infty} H_n(a) = A > 0 \qquad (A = \infty \text{ included}). \tag{6}$$

Then

$$\lim_{n\to\infty} H_n(pa+q) = pA+q \tag{7}$$

for all positive constants p, q, *if and only if*

$$\lim_{n\to\infty} G_n(a) = A\,. \tag{8}$$

The assertion is no longer true if $A = 0$.

2. For the proof of Theorem 2 we need the following

Theorem 3. *Let* $\{a_n\}$ $(n = 1, 2, \ldots)$ *be an infinite sequence of positive real numbers such that*

$$\lim_{n\to\infty} A_n(a) = \lim_{n\to\infty} G_n(a) = A < \infty\,. \tag{9}$$

Then, for all positive constants p *and* q,

$$\lim_{n\to\infty} H_n(pa+q) = pA+q\,.$$

Here we cannot include $q = 0$.

Before proving the theorem we state a lemma below.

Lemma 1. *If* $a_k \geqq 0$ $(k = 1, 2, \ldots, n)$, *then*

$$\sqrt[n]{(1+a_1)(1+a_2)\ldots(1+a_n)} \geqq 1 + G_n(a)\,, \tag{a}$$

$$\frac{1}{n}\sum_{k=1}^{n} \frac{a_k}{1+a_k} \leqq \frac{A_n(a)}{1+A_n(a)}\,. \tag{b}$$

Proof. Although these inequalities are known and proved easily by induction, we find it interesting to show another simple way. Let us put $a_k = \tan^2\theta_k$ in (a) and (b). Then, after simplification, we see that (a) and (b) are equivalent to

$$\sqrt[n]{\cos^2\theta_1 \cdot \cos^2\theta_2 \ldots \cos^2\theta_n} + \sqrt[n]{\sin^2\theta_1 \cdot \sin^2\theta_2 \ldots \sin^2\theta_n} \leqq 1 \tag{10}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} \cos^2\theta_k \geqq \left(\frac{1}{n}\sum_{k=1}^{n} \sec^2\theta_k\right)^{-1}, \tag{11}$$

respectively. Now (11) is self-evident and (10) follows from

$$\sqrt[n]{\prod_{k=1}^{n} \cos^2\theta_k} + \sqrt[n]{\prod_{k=1}^{n} \sin^2\theta_k} \leqq \frac{1}{n}\sum_{k=1}^{n} \cos^2\theta_k + \frac{1}{n}\sum_{k=1}^{n} \sin^2\theta_k = 1 .$$

Proof of Theorem 3. From (a) we have inequalities

$$1 + G_n(a) \leqq \left\{\prod_{k=1}^{n} (1 + a_k)\right\}^{1/n} \leqq 1 + A_n(a) ,$$

which imply together with (9) that

$$\lim_{n\to\infty} \left\{\prod_{k=1}^{n} (1 + a_k)\right\}^{1/n} = 1 + A . \tag{12}$$

Thus we obtain from (9) and (12),

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{a_k}{1+a_k} \geqq \liminf_{n\to\infty} \left(\prod_{k=1}^{n} \frac{a_k}{1+a_k}\right)^{1/n} = \frac{A}{1+A} .$$

On the other hand, from (9) and (b), we get

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{a_k}{1+a_k} \leqq \limsup_{n\to\infty} \frac{A_n(a)}{1+A_n(a)} = \frac{A}{1+A} .$$

Hence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{a_k}{1+a_k} = \frac{A}{1+A} , \tag{13}$$

i.e.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+a_k} = \frac{1}{1+A} .$$

The rest part of the theorem follows from the example (1).
Proof of Theorem 2. First we need an inequality due to Henrici [1] [2, Chap. 3].

Lemma 2. *If* $0 < a_k \leqq 1$ $(k = 1, 2, \ldots, n)$, *then*

$$\frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+a_k} \leqq \frac{1}{1+G_n(a)} . \tag{14}$$

Since we may assume, without loss of generality, $0 < a_n \leqq 1$ in Theorem 2, it follows from (7) and (14) (with $p = q = 1$) that

$$\limsup_{n\to\infty} G_n(a) \leqq A . \tag{15}$$

On the other hand, from (6) we have

$$\liminf_{n\to\infty} G_n(a) \geqq \liminf_{n\to\infty} H_n(a) = A ,$$

which implies together with (15) that

$$\lim_{n\to\infty} G_n(a) = A\,, \qquad (A = \infty \text{ being included})$$

i.e. the "only if" part of Theorem 2.
To prove the "if" part we shall make the substitution $a_n = 1/b_n$ in Theorem 3. Then (13) becomes

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+b_k} = \left(1 + \frac{1}{A}\right)^{-1},$$

which proves (7). That we cannot include $A = 0$ in this theorem may be seen from

$$a_n = 1\ (n \neq \nu^2)\,, \qquad \frac{1}{n}\ (n = \nu^2)\,, \tag{16}$$

in which case $\lim_{n\to\infty} G_n(a) = \lim_{n\to\infty} H_n(a) = 0$, but $\lim_{n\to\infty} H_n(pa+q) = 2$ for $p = q = 1$.

Proof of Theorem 1. Finally we deduce Theorem 1 from Theorem 2. In fact, by the substitution $a_n = 1/b_n$ in Theorem 2, we obtain the same assertion as Theorem 1 for b_n in place of a_n. That we cannot drop (3) in Theorem 1 may be seen also from (16) where $\lim_{n\to\infty} A_n(a) = 1$, $\lim_{n\to\infty} G_n(a) = 0$, while $\lim_{n\to\infty} H_n(pa+q) = p + q$ for all positive constants p, q. Also example (1) shows that $q = 0$ should be excluded in Theorem 1. On the other hand, the example $a_n = n$ (n even), 1 (n odd) shows that we have to exclude $A = \infty$ because in this case

$$\lim_{n\to\infty} A_n(a) = \lim_{n\to\infty} G_n(a) = \infty\,,$$

while

$$\lim_{n\to\infty} H_n(pa+q) = 2\,(p+q)$$

for all $p, q > 0$. Thus our proof of Theorem 1 is complete.

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REFERENCES

1 P. Henrici: Problem 245, Elem. Math. *11*, 112 (1956).
2 D. S. Mitrinovic: Analytic Inequalities. Springer, Berlin-Heidelberg-New York 1970.

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Die Eulersche Formel $\cos\varphi + i \cdot \sin\varphi = e^{i\varphi}$

Diese wichtige Formel wird üblicherweise mit Hilfe der Potenzreihen bewiesen. Es ist aber wünschenswert, sie schon möglichst früh zur Verfügung zu haben, und daher mag der folgende Beweis von Interesse sein, welcher lediglich einfache Tatsachen über komplexe Zahlen, trigonometrische Funktionen und Grenzwerte verwendet.

Für alle z sei $\exp z$ als Limes von $\left(1+\frac{z}{n}\right)^n$ definiert. Es ist

$$\cos\varphi + i\cdot\sin\varphi = \left(\cos\frac{\varphi}{n} + i\cdot\sin\frac{\varphi}{n}\right)^n = \left(1+\frac{i\varphi}{n}\right)^n (1-w)^n.$$

Der erste Faktor geht gegen $e^{i\varphi}$, und wir müssen zeigen, dass der zweite gegen 1 geht. In ihm ist mit $\alpha = \varphi/n$

$$w = \frac{(1-\cos\alpha) + i(\alpha - \sin\alpha)}{1+i\alpha}.$$

Wir verwenden

$$0 \leqq 1-\cos\alpha = 2\cdot\sin^2\frac{\alpha}{2} \leqq \frac{\alpha^2}{2}.$$

Von jetzt an sei $|\alpha| < 1$, d. h. $n > |\varphi|$. Wegen $|\alpha| \leqq |\tan\alpha|$ folgt

$$|\alpha - \sin\alpha| \leqq |\alpha|(1-\cos\alpha) \leqq \frac{|\alpha|^3}{2}$$

und damit

$$|w|^2 \leqq \frac{\alpha^4/4 + \alpha^6/4}{1+\alpha^2} \leqq \alpha^4 \quad \text{und schliesslich} \quad |w| \leqq \frac{\varphi^2}{n^2}.$$

Die letzte Abschätzung führt auf $\lim_{n\to\infty}(1-w)^n = 1$.
Für $n > \varphi^2$ ist nämlich

$$|(1-w)^n - 1| \leqq \sum_{k=1}^{\infty} |n w|^k = \frac{n|w|}{1-n|w|} \leqq \frac{\varphi^2}{n-\varphi^2}.$$

Man bemerkt noch, dass aus dem Beweis die Existenz des Grenzwerts $\lim_{n\to\infty}\left(1+\frac{i\varphi}{n}\right)^n$ folgt.

D. Laugwitz, Technische Hochschule Darmstadt

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