

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 39 (1984)
Heft: 2

Artikel: A remark on buffon's needle problem
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DOI: <https://doi.org/10.5169/seals-38015>

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A Remark on Buffon's Needle Problem

Let G_n be a grid in the Euclidean space \mathbb{R}^n with coordinates x_1, x_2, \dots, x_n determined by hyperplanes parallel to the hyperplanes with the equation $x_i = 0$ separated by a distance of $2L$.

In his generalization of Buffon's Needle Problem to n dimensions Stoka [2] gives an expression for the probability that a segment ω of length L which will be "thrown" in a random fashion into the \mathbb{R}^n cuts the grid G_n .

Let A_j be the event: the segment ω cuts a hyperplane of G_n parallel to the hyperplane with the equation $x_j = 0$. In [2] and [3] Stoka shows that the probability for the event A_j is

$$P(A_j) = \frac{\Gamma\left(\frac{n}{2}\right)}{(n-1) \Gamma\left(\frac{n-1}{2}\right) \sqrt{\pi}} \quad (j = 1, 2, \dots, n) \quad (1)$$

(formula (9) in [3] is a misprint) and finds for the estimator

$$\hat{P}_n = \frac{1}{nN} \sum_{j=1}^n (\text{number of times } A_j \text{ occurs in } N \text{ independent trials})$$

of $P(A_j)$ the variance

$$D^2(\hat{P}_n) = \frac{1}{nN} \theta_n \quad (2)$$

with

$$\begin{aligned} \theta_n &= \frac{(2n-1) \Gamma\left(\frac{n}{2}\right)}{(n-1) \Gamma\left(\frac{n-1}{2}\right) \sqrt{\pi}} - \frac{n \left[\Gamma\left(\frac{n}{2}\right) \right]^2}{(n-1)^2 \left[\Gamma\left(\frac{n-1}{2}\right) \right]^2 \pi} - n + 1 \\ &+ \frac{2^{n-2} (n-1) \Gamma\left(\frac{n}{2}\right)}{\pi^{3/2} (n-2)!} \left[\left(\pi + \frac{1}{2n} \right) \Gamma\left(\frac{n-1}{2}\right) - \frac{2\sqrt{\pi}}{n-1} \Gamma\left(\frac{n}{2}\right) \right] \end{aligned}$$

([3], formula (15)). On the other hand he calculates for the estimator $\hat{P}_1 = M^{-1}$ (number of times A_1 occurs in M independent trials) of $P(A_j)$ the variance

$$D^2(\hat{P}_1) = \frac{1}{M} \frac{\Gamma\left(\frac{n}{2}\right)}{(n-1) \Gamma\left(\frac{n-1}{2}\right) \sqrt{\pi}} \left[1 - \frac{\Gamma\left(\frac{n}{2}\right)}{(n-1) \Gamma\left(\frac{n-1}{2}\right) \sqrt{\pi}} \right] \quad (3)$$

([3], formula (16)). For the case $n = 2$ compare Schuster [1]. If we set both variances (2) and (3) equal we receive

$$M = \sigma(n) n N \quad (4)$$

with

$$\sigma(n) = \frac{\Gamma\left(\frac{n}{2}\right) \left[(n-1) \Gamma\left(\frac{n-1}{2}\right) \sqrt{\pi} - \Gamma\left(\frac{n}{2}\right)\right]}{(n-1)^2 \left[\Gamma\left(\frac{n-1}{2}\right)\right]^2 \pi \theta_n}. \quad (5)$$

Thus, N independent trials in G_n give the same information about the probability (1) as $\sigma(n) n N$ independent trials in a grid determined by hyperplanes parallel to the hyperplane with the equation $x_1 = 0$. Schuster [1] finds $\sigma(2) \approx 1.1114$. Stoka [3] calculates $\sigma(3) \approx 1.1121$ and $\sigma(4) \approx 1.1039$ and conjectures that $\lim_{n \rightarrow \infty} \sigma(n) = 1$. In this note we give an asymptotic expression for $\sigma(n)$ and show that Stoka's conjecture is true.

Theorem. *It holds*

$$\sigma(n) = \frac{\pi n! - 2^{n-2} n \left[\Gamma\left(\frac{n}{2}\right)\right]^2}{\pi n! - 2^{n-2} n^2 \left[\Gamma\left(\frac{n}{2}\right)\right]^2 + 2^{n-3} (n-1)^3 \left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}, \quad (6)$$

$$\sigma(n) \cong \frac{\sqrt{2\pi} - n^{-1/2} \left(\frac{n-2}{n}\right)^{n-1} e^2}{\sqrt{2\pi} - n^{1/2} \left(\frac{n-2}{n}\right)^{n-1} e^2 + n^{1/2} \left(\frac{n-1}{n}\right)^3 \left(\frac{n-3}{n}\right)^{n-2} e^3} \quad (n \rightarrow \infty) \quad (7)$$

and

$$\lim_{n \rightarrow \infty} \sigma(n) = 1. \quad (8)$$

Proof: First we simplify the expression (5). For abbreviation we set

$a = a_n = \Gamma\left(\frac{n-1}{2}\right)$ and $b = b_n = \Gamma\left(\frac{n}{2}\right)$. Then by (5)

$$\begin{aligned} \sigma(n) &= \frac{(n-1) \sqrt{\pi} a b - b^2}{(n-1)^2 \pi \left\{ \frac{(2n-1) a b}{(n-1) \sqrt{\pi}} - \frac{n b^2}{(n-1)^2 \pi} - (n-1) a^2 + \right.} \\ &\quad \left. + \frac{2^{n-2} (n-1) \left(\pi + \frac{1}{2n}\right) a^3 b}{\pi^{3/2} (n-2)!} - \frac{2^{n-1} a^2 b^2}{\pi (n-2)!} \right\}}. \end{aligned}$$

The formula

$$\Gamma(x) \Gamma\left(x + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2x-1}} \Gamma(2x)$$

gives us

$$ab = \frac{\sqrt{\pi}}{2^{n-2}} (n-2)!$$

and therefore

$$\begin{aligned} \sigma(n) &= \frac{2^{-(n-2)} \pi (n-1)! - b^2}{2^{-(n-2)} \pi (n-1)! (2n-1) - n b^2 + \frac{(n-1)^3}{2n} a^2 - 2^{-(n-3)} \pi (n-1)! (n-1)} \\ &= \frac{\pi n! - 2^{n-2} n b^2}{\pi n! - 2^{n-2} n^2 b^2 + 2^{n-3} (n-1)^3 a^2} \end{aligned}$$

which is (6).

By Stirling's formula for the Gamma-function it holds

$$\log \Gamma(x+1) = \left(x + \frac{1}{2}\right) \log x - x + c + O\left(\frac{1}{x}\right) \quad (x \rightarrow +\infty)$$

with

$$c = \frac{1}{2} \log(2\pi).$$

This implies

$$\begin{aligned} \sigma(n) &\cong \frac{\pi n^{n+1/2} e^{-n+c} - 2^{n-2} n \left(\frac{n-2}{2}\right)^{n-1} e^{-(n-2)+2c}}{\pi n^{n+1/2} e^{-n+c} - 2^{n-2} n^2 \left(\frac{n-2}{2}\right)^{n-1} e^{-(n-2)+2c} + 2^{n-3} (n-1)^3 \left(\frac{n-3}{2}\right)^{n-2} e^{-(n-3)+2c}} \\ &= \frac{\sqrt{2\pi} - n^{-1/2} \left(\frac{n-2}{n}\right)^{n-1} e^2}{\sqrt{2\pi} - n^{1/2} \left(\frac{n-2}{n}\right)^{n-1} e^2 + n^{1/2} \left(\frac{n-1}{n}\right)^3 \left(\frac{n-3}{n}\right)^{n-2} e^3} \end{aligned}$$

which is (7).

Evidently the numerator of (7) converges to $\sqrt{2\pi}$ for $n \rightarrow \infty$. It is sufficient to show that

$$d = \lim_{n \rightarrow \infty} n^{1/2} \left[\left(\frac{n-2}{n} \right)^{n-1} - \left(\frac{n-1}{n} \right)^3 \left(\frac{n-3}{n} \right)^{n-2} e \right] = 0.$$

Now

$$\begin{aligned} |d| &\leq \lim_{n \rightarrow \infty} n^{1/2} \left(\frac{n-1}{n} \right)^3 \left(\frac{n-3}{n} \right)^{n-2} \\ &\quad \cdot \left\{ \left| \left(\frac{n}{n-1} \right)^3 \left(\frac{n-2}{n} \right) - 1 \right| \left(\frac{n-2}{n-3} \right)^{n-2} + \left| \left(\frac{n-2}{n-3} \right)^{n-2} - e \right| \right\}. \end{aligned}$$

Because of

$$\left(\frac{n}{n-1} \right)^3 \frac{n-2}{n} - 1 = 0 \left(\frac{1}{n} \right) \quad (n \rightarrow \infty)$$

it remains to show that

$$\left(1 + \frac{1}{n} \right)^{n+1} = e + 0 \left(\frac{1}{n} \right) \quad (n \rightarrow \infty).$$

To see this we set $x_i = i/n$ ($i = 0, 1, \dots, n$) and Taylor's formula gives us

$$e^{x_i} = \left(1 + \frac{1}{n} \right) e^{x_{i-1}} + \frac{1}{2n^2} e^{\xi_i} \quad \text{with} \quad x_{i-1} < \xi_i < x_i.$$

This implies

$$\begin{aligned} e = e^{x_n} &= \left(1 + \frac{1}{n} \right)^n e^{x_0} + \frac{1}{2n^2} \sum_{i=1}^n \left(1 + \frac{1}{n} \right)^{n-i} e^{\xi_i} \\ &= \left(1 + \frac{1}{n} \right)^n + 0 \left(\frac{1}{n} \right) \quad (n \rightarrow \infty). \end{aligned}$$

Thus the Theorem is proved.

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