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A Remark on Buffon's Needle Problem

Let G_n be a grid in the Euclidean space $\mathbb{R}^n$ with coordinates $x_1, x_2, \ldots, x_n$ determined by hyperplanes parallel to the hyperplanes with the equation $x_i = 0$ separated by a distance of $2L$.
In his generalization of Buffon's Needle Problem to n dimensions Stoka [2] gives an expression for the probability that a segment ω of length L which will be "thrown" in a random fashion into the $\mathbb{R}^n$ cuts the grid G_n.
Let A_j be the event: the segment ω cuts a hyperplane of G_n parallel to the hyperplane with the equation $x_j = 0$. In [2] and [3] Stoka shows that the probability for the event A_j is

$$P(A_j) = \frac{\Gamma\left(\frac{n}{2}\right)}{(n-1)\,\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}} \qquad (j = 1, 2, \ldots, n) \tag{1}$$

(formula (9) in [3] is a misprint) and finds for the estimator

$$\hat{P}_n = \frac{1}{nN} \sum_{j=1}^{n} \text{(number of times } A_j \text{ occurs in } N \text{ independent trials)}$$

of $P(A_j)$ the variance

$$D^2(\hat{P}_n) = \frac{1}{nN}\,\theta_n \tag{2}$$

with

$$\theta_n = \frac{(2n-1)\,\Gamma\left(\frac{n}{2}\right)}{(n-1)\,\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}} - \frac{n\left[\Gamma\left(\frac{n}{2}\right)\right]^2}{(n-1)^2\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2 \pi} - n + 1$$

$$+ \frac{2^{n-2}(n-1)\,\Gamma\left(\frac{n}{2}\right)}{\pi^{3/2}(n-2)!}\left[\left(\pi + \frac{1}{2n}\right)\Gamma\left(\frac{n-1}{2}\right) - \frac{2\sqrt{\pi}}{n-1}\,\Gamma\left(\frac{n}{2}\right)\right]$$

([3], formula (15)). On the other hand he calculates for the estimator $\hat{P}_1 = M^{-1}$ (number of times A_1 occurs in M independent trials) of $P(A_j)$ the variance

$$D^2(\hat{P}_1) = \frac{1}{M}\,\frac{\Gamma\left(\frac{n}{2}\right)}{(n-1)\,\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}}\left[1 - \frac{\Gamma\left(\frac{n}{2}\right)}{(n-1)\,\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}}\right] \tag{3}$$

([3], formula (16)). For the case $n=2$ compare Schuster [1]. If we set both variances (2) and (3) equal we receive

$$M=\sigma(n)\, n N \tag{4}$$

with

$$\sigma(n)=\frac{\Gamma\left(\frac{n}{2}\right)\left[(n-1)\,\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}-\Gamma\left(\frac{n}{2}\right)\right]}{(n-1)^2\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2\pi\,\theta_n}. \tag{5}$$

Thus, N independent trials in G_n give the same information about the probability (1) as $\sigma(n)\, n N$ independent trials in a grid determined by hyperplanes parallel to the hyperplane with the equation $x_1=0$. Schuster [1] finds $\sigma(2)\approx 1.1114$. Stoka [3] calculates $\sigma(3)\approx 1.1121$ and $\sigma(4)\approx 1.1039$ and conjectures that $\lim_{n\to\infty}\sigma(n)=1$. In this note we give an asymptotic expression for $\sigma(n)$ and show that Stoka's conjecture is true.

Theorem. *It holds*

$$\sigma(n)=\frac{\pi\, n!-2^{n-2}\, n\left[\Gamma\left(\frac{n}{2}\right)\right]^2}{\pi\, n!-2^{n-2}\, n^2\left[\Gamma\left(\frac{n}{2}\right)\right]^2+2^{n-3}(n-1)^3\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}, \tag{6}$$

$$\sigma(n)\cong\frac{\sqrt{2\pi}-n^{-1/2}\left(\frac{n-2}{n}\right)^{n-1}e^2}{\sqrt{2\pi}-n^{1/2}\left(\frac{n-2}{n}\right)^{n-1}e^2+n^{1/2}\left(\frac{n-1}{n}\right)^3\left(\frac{n-3}{n}\right)^{n-2}e^3}\quad (n\to\infty) \tag{7}$$

and

$$\lim_{n\to\infty}\sigma(n)=1. \tag{8}$$

Proof: First we simplify the expression (5). For abbreviation we set $a=a_n=\Gamma\left(\frac{n-1}{2}\right)$ and $b=b_n=\Gamma\left(\frac{n}{2}\right)$. Then by (5)

$$\sigma(n)=\frac{(n-1)\sqrt{\pi}\, a\, b-b^2}{(n-1)^2\pi\left\{\frac{(2n-1)\, a\, b}{(n-1)\sqrt{\pi}}-\frac{n\, b^2}{(n-1)^2\pi}-(n-1)\, a^2+\frac{2^{n-2}(n-1)\left(\pi+\frac{1}{2n}\right)a^3 b}{\pi^{3/2}(n-2)!}-\frac{2^{n-1}a^2 b^2}{\pi\,(n-2)!}\right\}}.$$

The formula

$$\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2x-1}}\,\Gamma(2x)$$

gives us

$$a\,b=\frac{\sqrt{\pi}}{2^{n-2}}\,(n-2)!$$

and therefore

$$\sigma(n)=\frac{2^{-(n-2)}\,\pi\,(n-1)!-b^2}{2^{-(n-2)}\,\pi\,(n-1)!\,(2n-1)-n\,b^2+\dfrac{(n-1)^3}{2n}\,a^2-2^{-(n-3)}\,\pi\,(n-1)!\,(n-1)}$$

$$=\frac{\pi\,n!-2^{n-2}\,n\,b^2}{\pi\,n!-2^{n-2}\,n^2\,b^2+2^{n-3}\,(n-1)^3\,a^2}$$

which is (6).

By Stirling's formula for the Gamma-function it holds

$$\log\Gamma(x+1)=\left(x+\frac{1}{2}\right)\log x-x+c+0\left(\frac{1}{x}\right)\qquad(x\to+\infty)$$

with

$$c=\frac{1}{2}\log(2\pi)\,.$$

This implies

$$\sigma(n)\cong\frac{\pi\,n^{n+1/2}\,e^{-n+c}-2^{n-2}\,n\left(\dfrac{n-2}{2}\right)^{n-1}e^{-(n-2)+2c}}{\pi\,n^{n+1/2}\,e^{-n+c}-2^{n-2}\,n^2\left(\dfrac{n-2}{2}\right)^{n-1}e^{-(n-2)+2c}+2^{n-3}\,(n-1)^3\left(\dfrac{n-3}{2}\right)^{n-2}e^{-(n-3)+2c}}$$

$$=\frac{\sqrt{2\pi}-n^{-1/2}\left(\dfrac{n-2}{n}\right)^{n-1}e^2}{\sqrt{2\pi}-n^{1/2}\left(\dfrac{n-2}{n}\right)^{n-1}e^2+n^{1/2}\left(\dfrac{n-1}{n}\right)^3\left(\dfrac{n-3}{n}\right)^{n-2}e^3}$$

which is (7).

Evidently the numerator of (7) converges to $\sqrt{2\pi}$ for $n \to \infty$. It is sufficient to show that

$$d = \lim_{n\to\infty} n^{1/2}\left[\left(\frac{n-2}{n}\right)^{n-1} - \left(\frac{n-1}{n}\right)^3 \left(\frac{n-3}{n}\right)^{n-2} e\right] = 0\,.$$

Now

$$|d| \leqq \lim_{n\to\infty} n^{1/2}\left(\frac{n-1}{n}\right)^3\left(\frac{n-3}{n}\right)^{n-2}$$
$$\cdot\left\{\left|\left(\frac{n}{n-1}\right)^3\left(\frac{n-2}{n}\right)-1\right|\left(\frac{n-2}{n-3}\right)^{n-2} + \left|\left(\frac{n-2}{n-3}\right)^{n-2} - e\right|\right\}.$$

Because of

$$\left(\frac{n}{n-1}\right)^3 \frac{n-2}{n} - 1 = 0\left(\frac{1}{n}\right) \quad (n \to \infty)$$

it remains to show that

$$\left(1+\frac{1}{n}\right)^{n+1} = e + 0\left(\frac{1}{n}\right) \quad (n \to \infty)\,.$$

To see this we set $x_i = i/n$ $(i = 0, 1, \ldots, n)$ and Taylor's formula gives us

$$e^{x_i} = \left(1+\frac{1}{n}\right) e^{x_{i-1}} + \frac{1}{2n^2} e^{\xi_i} \quad \text{with} \quad x_{i-1} < \xi_i < x_i\,.$$

This implies

$$e = e^{x_n} = \left(1+\frac{1}{n}\right)^n e^{x_0} + \frac{1}{2n^2}\sum_{i=1}^{n}\left(1+\frac{1}{n}\right)^{n-i} e^{\xi_i}$$
$$= \left(1+\frac{1}{n}\right)^n + 0\left(\frac{1}{n}\right) \quad (n \to \infty)\,.$$

Thus the Theorem is proved.

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